

On Singular Perturbation Problems with Several Turning Points

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§1. Introduction. Let Ω be an open bounded domain in R^m with smooth boundary $\partial\Omega$. We shall study the asymptotic behavior as $\varepsilon \rightarrow 0$ of the solutions of the first boundary-value problem

$$(1.1) \quad L_\varepsilon u = \varepsilon \left(\sum_{i,j=1}^m a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^m a_i(x) u_{x_i} \right) + \sum_{i=1}^m b_i(x) u_{x_i} = 0 \quad \text{in } \Omega$$

$$(1.2) \quad u|_{\partial\Omega} = \varphi(x).$$

Here $x = (x_1, x_2, \dots, x_m) \in R^m$ and

$$(1.3) \quad \sum_{i,j=1}^m a_{ij} \xi_i \xi_j \geq \nu \sum_{i=1}^m \xi_i^2, \quad (\nu > 0).$$

A problem of this type arises in the context of the motion of a Brownian particle in a force field. The escape of the particle over a potential barrier in higher dimensions gives rise to singular perturbations in the elliptic boundary-value problem, as in (1.1) and (1.2).

For examples and applications in physics, see [12].

It is natural to expect that the limit function (if it exists) should satisfy the degenerate equation

$$(1.4) \quad \sum b_i U_{x_i} = 0.$$

The question arises as to what additional conditions would uniquely determine the solution of (1.4). The answer depends on the behavior of the characteristic curves of (1.4), i.e., the integral curves of the system

$$(1.5) \quad \frac{dx_i}{dt} = b_i,$$

which are also called its trajectories. The problem has been well studied for the case in which the vector field $b = (b_1, \dots, b_m)$ is regular, in the sense that every trajectory ℓ that enters the domain at some point $P_1 \in \partial\Omega$ leaves $\bar{\Omega}$ at some $P_2 \in \partial\Omega$. Then $u_\varepsilon(x) \rightarrow \varphi(P_2)$ at every point $x \in \ell$, and near the points where