

Power Series Rings with Finite Krull Dimension

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1. Introduction. Let R be a commutative ring with identity and denote by $\dim R$ the Krull dimension of R . It is well known that if $\dim R = m$ then $m + 1 \leq \dim R[x] \leq 2m + 1$ [14, Theorem 2]. The author has shown in [2] that these bounds do not hold for the power series ring $R[[x]]$. Specifically, if R fails to have the SFT-property then $R[[x]]$ has infinite Krull dimension [2, Theorem 1]. An ideal A of the ring R is called an *SFT-ideal* (an ideal of *strong finite type*) provided there exists a positive integer k and a finitely generated ideal B such that $B \subseteq A$ and $a^k \in B$ for each $a \in A$. The ring R has the *SFT-property* if each ideal of R is an SFT-ideal.

The basic unsolved problem that motivates this paper is the problem of determining the possibilities for $\dim R[[x]]$ or, more generally, the possibilities for $\dim R[[x_1, \dots, x_n]]$ when these quantities are finite. The case when R is Noetherian is well known; that is, if R is Noetherian and $\dim R = m$ then $\dim R[[x_1, \dots, x_n]] = m + n$ for each n . If R is a Prüfer domain then, as in the Noetherian case, $\dim R[x_1, \dots, x_n] = m + n$. It thus seems reasonable to ask whether $\dim R[[x_1, \dots, x_n]] = m + n$ for each n if R is a Prüfer domain with the SFT-property. This is the case if $\dim R = 1$, for then R is Dedekind domain [3, page 4]. The results of [3] further suggest an affirmative answer for it is shown in [3, Theorem 3.8] that $\dim R[[x]] = m + 1$ and that, as in the polynomial ring case (cf. [8, Theorem 19.15]), if P is a prime ideal of R then the only prime ideals of $R[[x]]$ contained in $P[[x]]$ are of the form $Q[[x]]$ for some prime ideal Q of R [3, Corollary 3.6]. The author has shown in [5], however, that if V is a discrete valuation ring with $\dim V = m$ (a finite-dimensional valuation ring has the SFT-property if and only if it is discrete (cf. Proposition 2.1(iii))) then $\dim V[[x_1, \dots, x_n]] = mn + 1$. In this paper we extend this equality, in Theorem 3.6, to an arbitrary Prüfer domain R with the SFT-property. The proof shows that if Q and P are prime ideals of R such that $Q \subset P$ and $\text{rank}(P/Q) = 1$, then for each n

$$\text{rank}(P[[x_1, \dots, x_n]]/Q[[x_1, \dots, x_n]]) = 1$$

if P is a maximal ideal but is n if P is nonmaximal. The analogue for polynomial rings cannot occur; that is, if P and Q are prime ideals of any ring R such that