

A Minimization Problem and the Regularity of Solutions in the Presence of a Free Boundary

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Introduction. The purpose of this paper is to obtain a sharp interior regularity estimate for a function $u(x)$ satisfying:

$$J(u) = \int_G \left(\frac{|\nabla u|^2}{2} + u^\gamma \right) dx = \min_{w \in \mathbf{K}} J(w), \quad u \in \mathbf{K}.$$

Here γ is a fixed constant with $0 < \gamma < 1$, G is an open bounded set in \mathbf{R}^n with ∂G locally a Lipschitz graph, and $\mathbf{K} = \{w : w \in H^1(G), w \geq 0, w = u_0|_{\partial G}\}$ where u_0 is a fixed element in $H^1(G)$ with $u_0 \geq 0$.

To be precise we will delineate a particular representative $u(x)$, and show that for each compact subset $H \subset G$, $u(x) \in C^{1,\beta-1}(H)$ with $\beta \equiv 2/(2 - \gamma)$.

Moreover we shall see that $u(x)$ will be a solution to the following free boundary problem:

$$(0.1) \quad \begin{array}{ll} \Delta u = \gamma u^{\gamma-1} & \text{in } \{x : u(x) > 0\} \\ u = 0, \nabla u = 0 & \text{on } \partial\{u > 0\} - \partial G, \\ u = u_0 & \text{on } \partial G. \end{array}$$

Two similar minimization problems (each with an associated free boundary problem) have recently been studied by H. Alt and L. Caffarelli.

In [5] $u(x)$ were considered that satisfied:

$$J^1(u) = \int_G (|\nabla u|^2 + u) dx = \min_{w \in \mathbf{K}} J^1(w), \quad u \in \mathbf{K},$$

where \mathbf{K} is defined as above.

In [1] $u(x)$ were considered that satisfied:

$$J^0(u) = \int_G (|\nabla u|^2 + \chi_{\{u>0\}}) dx = \min_{w \in \mathbf{K}} J^0(w), \quad u \in \mathbf{K},$$

again with \mathbf{K} as above, where $\chi_{\{u>0\}}$ represents the characteristic function of $\{x : u(x) > 0\}$.