

# *On the Regularity of Boundaries of Sets Minimizing Perimeter with a Volume Constraint*

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## **Introduction**

The regularity of the reduced boundary  $\partial^*E$  of a set  $E$ , minimizing perimeter in an open set  $\Omega \subset \mathbf{R}^n$  ( $n \geq 2$ ), was proved by E. De Giorgi in 1960 (see [4] or [6]); later, M. Miranda [18] obtained the following estimate on the measure of the singular points:  $H_{n-1}[(\partial E - \partial^*E) \cap \Omega] = 0$ . This result was then improved by H. Federer [8], who showed that actually  $H_s[(\partial E - \partial^*E) \cap \Omega] = 0$ , for every real  $s > n - 8$ ; this is the best result one may expect, in view of the well-known example of the ‘‘Simons cone’’ which is area-minimizing in  $\mathbf{R}^8$  but singular at  $x = 0$  (see [3]). Analogous results were obtained by U. Massari for boundaries of sets having prescribed mean curvature in  $\Omega$ , using essentially the same method as in De Giorgi’s paper (see [17]).

The constraint usually imposed on the solution  $E$  is some kind of boundary condition on  $\partial\Omega$ , which does not affect its interior regularity. However, it appears appropriate for several applications to impose on the solution a volume constraint, too; the first example in this direction is the *isoperimetric problem*: to find a set with minimal perimeter and a given volume. As everybody knows, in this case the solution can be found explicitly (see e.g. [5] and [13]), so that any question about its regularity is actually superfluous.

Further examples of minima of area-like functionals, subject to volume constraints, are often encountered in the field of Capillarity Theory: liquid drops, resting on or hanging from a given surface, are the simplest ones among them. They have recently been studied by the authors in [14], [15] and [16], in the case of a horizontal plane as supporting surface. One knows that in this case the solution is axially symmetric [14], so that its boundary can be locally described (except perhaps near points lying on the axis of symmetry itself) as a graph of a suitable function, which turns out to solve a similar problem.

Now, it is well known (see [9] and [10]) that if a function  $f \in BV(\Omega)$  minimizes an area-like functional  $J(f)$ , with the constraints  $f \geq \psi$  in  $\Omega$  and  $\int_{\Omega} (f - \psi) dx = v$  ( $\psi$  and  $v$  given), then there exists a multiplier  $\lambda \in \mathbf{R}$  such that  $f$  min-