

# *The Lattice of Operator Ranges of a von Neumann Algebra*

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*Dedicated to Professor Paul R. Halmos on his 65th birthday*

**1. Introduction.** By means of a simple elegant construction, P. R. Halmos shows in [10, page 26] how the sum of two closed subspaces of a Hilbert space may turn out to be nonclosed. The same construction produces a simple proof of the theorem of Mackey [12] and Dixmier [3] that the closed subspaces generate the lattice of operator ranges and leads to a criterion for when this is true of a von Neumann algebra. I am greatly indebted to Peter Fillmore for having called my attention to his results in [5] which led to a substantial shortening of the proof of Corollary 3.3 and to the elimination of a separability hypothesis.

Let  $\mathfrak{A}$  be a von Neumann algebra of operators on a Hilbert space  $\mathcal{H}$ . Let  $\mathbf{L}_c$  be the set of (closed) subspaces that are ranges of operators in  $\mathfrak{A}$ , or  $\mathbf{L}_c$  may be identified with the projections in  $\mathfrak{A}$ . Then it is well known that  $\mathbf{L}_c$  is a lattice (indeed a complete lattice) under the operations of closed span and intersection, and  $\mathbf{L}_c$  is the lattice of all subspaces invariant under every operator in the commutant  $\mathfrak{A}'$  of  $\mathfrak{A}$ . We are interested in analogous properties for operator ranges.

Let  $\mathbf{L}$  be the collection of ranges of operators in  $\mathfrak{A}$ . Mackey [12] and Dixmier [3] proved in case  $\mathfrak{A}$  is the set  $\mathfrak{B}(\mathcal{H})$  of all operators that  $\mathbf{L}$  is a lattice under the operation of sum (without closure) and intersection, and Fillmore and Williams [7] observed that this is true of every von Neumann algebra. Mackey and Dixmier also proved that in the case of  $\mathfrak{B}(\mathcal{H})$  the lattice  $\mathbf{L}$  is generated by  $\mathbf{L}_c$ . Our purpose is to show this is true for all von Neumann algebras having no infinite dimensional commutative summand. These are precisely the von Neumann algebras that were shown to be algebraically generated by their projections by Fillmore and Topping [6].

It is immediate that the members of  $\mathbf{L}$  are left invariant by the operators in  $\mathfrak{A}'$ . Whether  $\mathbf{L}$  contains all operator ranges left invariant by all members of  $\mathfrak{A}'$  is an interesting open question of Dixmier. Bunce [1] has pointed out that a proof of Foias [7] shows the answer to this question is affirmative if every continuous homomorphism of  $\mathfrak{A}'$  into  $\mathfrak{B}(\mathcal{H})$  is similar to a  $*$ -homomorphism. Bunce shows that the converse assertion holds for continuous homomorphisms with cyclic vectors. His proof depends on his result that cyclic bounded homomorphisms of  $C^*$ -algebras into  $\mathfrak{B}(\mathcal{H})$  are quasi-similar to  $*$ -homomorphisms. Christensen [2] has