

# *An Extension of Alexander's Theorem on Proper Self Mappings of the Ball in $\mathbf{C}^n$*

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H. Alexander [1] proved that proper holomorphic self mappings of the unit ball in  $\mathbf{C}^n$  with  $n > 1$  are automorphisms. It is proved herein that a proper mapping of a bounded strictly pseudoconvex domain with real analytic boundary onto a smooth bounded Reinhardt domain must be biholomorphic. The new approach to this problem provides a simple and elementary proof of Alexander's original theorem.

Let  $D_1$  be a smooth bounded strictly pseudoconvex domain with real analytic boundary and let  $D_2$  be a smooth bounded pseudoconvex complete Reinhardt domain, both contained in  $\mathbf{C}^n$ . Suppose that  $f$  is a proper holomorphic mapping of  $D_1$  onto  $D_2$ . We shall prove that  $f$  is actually biholomorphic and that  $f$  extends to be a biholomorphic mapping between two larger domains  $G_1$  and  $G_2$  containing  $\bar{D}_1$  and  $\bar{D}_2$ , respectively. From this, it follows that  $D_2$  must also be a strictly pseudoconvex domain with real analytic boundary.

The key ingredients of the proof of this result are:

**Fact 1.** Let  $P_1$  denote the Bergman orthogonal projection of  $L^2(D_1)$  onto its subspace of holomorphic functions. If  $\phi \in C_0^\infty(D_1)$ , then  $P_1\phi$  extends to be holomorphic in a neighborhood of  $\bar{D}_1$ .

**Fact 2.** Let  $P_2$  denote the Bergman projection associated to  $D_2$ . For each monomial  $z^\alpha$ , there is a function  $\phi_\alpha$  in  $C_0^\infty(D_2)$  such that  $P_2\phi_\alpha = z^\alpha$ .

**Fact 3.** Let  $u = \text{Det}[f']$ . If  $\phi \in L^2(D_2)$ , then  $u \cdot (\phi \circ f) \in L^2(D_1)$  and  $P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2\phi) \circ f)$ .

Fact 3 is proved in [2], [3]. Fact 1 is a direct consequence of the analytic hypoellipticity of the  $\bar{\partial}$ -Neumann operator on strictly pseudoconvex domains with real analytic boundary (Tartakoff [7], Trèves [8]) and Kohn's formula  $P_1 = I - \bar{\partial}^*N\bar{\partial}$  which relates  $P_1$  to the  $\bar{\partial}$ -Neumann operator  $N$  for  $D_1$ .

Fact 2 is a simple consequence of the formula

$$\langle h, z^\alpha \rangle_{L^2(D_2)} = c_\alpha \frac{\partial^\alpha h}{\partial z^\alpha}(0)$$

which holds in complete Reinhardt domains because the monomials form a com-