

# *Flatness of Riemannian Metrics Outside Compact Sets*

TH. HASANIS & D. KOUTROFIOTIS

In a recent article [4], D. Elerath proved the following interesting result: Let  $M$  be a complete, open, nonnegatively curved 3-manifold with a point at which all the sectional curvatures are positive; if, in addition,  $M$  contains a pole, then it cannot be flat off any compact set. This result poses of course the question of its validity for higher-dimensional manifolds and for other curvature restrictions. We shall show here that higher-dimensional versions are indeed valid, even local ones, also for nonpositively curved manifolds. The method we employ is essentially the same as Elerath's: instead of the Gauss-Bonnet formula, we use the generalized Gauss-Bonnet formula.

As noted by Elerath and the reviewer in *MR 80e: 53041*, we have counterexamples of the above result in dimension 2: take a circular cone in  $\mathbf{R}^3$ , and replace a neighbourhood of its vertex by a smooth, convex cap. Here is an example, due to L. Nirenberg, of a complete, open, nonpositively curved 2-manifold with points of negative curvature, which is flat off a compact set: take a smooth, convex function  $y = f(x)$ , defined on  $\mathbf{R}$ , with  $f(0) = 0$ ,  $f'(0) = 0$ ,  $f''(0) > 0$ , which is linear for  $|x| \geq 1$ . The surface  $z = f(x) - f(y)$  in  $\mathbf{R}^3$  has curvature  $K \leq 0$ ,  $K = -[f''(0)]^2 < 0$  at the origin, and is flat at all points  $(x, y)$  for which  $\max\{|x|, |y|\} \geq 1$ . It would be interesting to have counterexamples in even dimensions greater than 2 as well.

We begin with terminology and some auxiliary propositions. Let  $M$  be a smooth,  $n$ -dimensional, Riemannian manifold, and let  $B_s(p_0)$  be an open ball centered at  $p_0$  in  $M^n$ , which is contained in a normal neighbourhood of  $p_0$ . Its boundary  $\partial B_s(p_0) = S_s(p_0)$ , a so-called *geodesic hypersphere*, is a compact, orientable hypersurface, topologically equivalent to the sphere  $S^{n-1}$ . We wish to give sufficient conditions for the second fundamental form of  $S_s(p_0)$  to be semidefinite (that is, not indefinite). If  $d$  denotes the Riemannian distance in  $M$ , we define the smooth function  $g: M \rightarrow R$  by  $g(p) = (1/2)[d(p_0, p)]^2$ ; its Hessian  $\nabla^2 g$  is a 2-covariant tensor defined by  $\nabla^2 g(X, Y) = \langle \nabla_X \text{grad } g, Y \rangle$ . Let  $p \in S_s(p_0)$ , and denote by  $\gamma$  the unit-speed geodesic with  $\gamma(0) = p_0$  and  $\gamma(s) = p$ . The unit vector  $\gamma'(s)$  is what we shall call the *outer normal* of  $S_s(p_0)$  at  $p$ . Let  $A_{\gamma'(s)}$  be the shape operator of  $S_s(p_0)$  at  $p$  in the direction  $\gamma'(s)$ , that is, for  $X \in T_p S_s(p_0)$ , we have by definition  $\nabla_X N = -A_{\gamma'(s)} X$ , where  $N$  stands for the unit-normal field to  $S_s(p_0)$  com-