

# The Hardy-Littlewood Maximal Function on $L(p, 1)$

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The Hardy-Littlewood maximal function of a locally integrable function,  $f$ , is

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(t)| dt,$$

the supremum taken over all cubes,  $Q$ , containing  $x$ . In a previous paper [1], we considered norm inequalities of the form

$$(1) \quad \|Mf\|_{p\bar{q}} \leq C \|f\|_{pq}, \quad 1 < p < \infty, 1 \leq q \leq \bar{q} \leq \infty,$$

where

$$\|f\|_{pq} = \begin{cases} \left( q \int_0^\infty w(\{x \in \mathbf{R}^n : |f(x)| > y\})^{q/p} y^{q-1} dy \right)^{1/q}, & 1 \leq q < \infty \\ \sup_{y>0} y w(\{x \in \mathbf{R}^n : |f(x)| > y\})^{1/p}, & q = \infty \end{cases}$$

is a norm on the weighted Lorentz space  $L(p, q) = L(p, q; w(x) dx)$  for  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , and  $w$  non-negative and locally integrable.

Recall that  $w \in A_p$ ,  $1 < p < \infty$ , if there is a constant  $C$  so that

$$\left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p/p'} \leq C$$

for all cubes  $Q \subset \mathbf{R}^n$ , and  $w \in A(p, 1)$  if for any cube  $Q$  and measurable set  $E \subset Q$ ,

$$\frac{|E|}{|Q|} \leq C \left[ \frac{w(E)}{w(Q)} \right]^{1/p}.$$

In [1], it was shown that, for  $q > 1$ , (1) holds if and only if  $w \in A_p$  and, for  $q = 1$  and  $\bar{q} = \infty$ , (1) is equivalent to  $w \in A(p, 1)$ . Although it was also proved that  $w \in A_p$  implies (1) with  $\bar{q} = q$ , the strongest possible inequality, the equivalence when  $q = 1$  and  $\bar{q} < \infty$ , was unresolved. It is interesting that for these