

# On a Theorem of Burago-Toponogov

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**1. Introduction.** For a compact connected Riemannian manifold  $M$  the injectivity radius  $i(M)$  of  $M$  is defined by

$$i(M) := \sup\{r > 0 : \exp|_{B_r(o_m)} \text{ is a diffeomorphism for all } m \in M\},$$

where  $\exp$  denotes the exponential mapping and  $B_r(o_m)$  open  $r$ -ball in the tangent space  $T_m(M)$  centered at origin  $o_m$ . Then  $i(M)$  is positive and given by the minimum of half the length of the shortest (simple) closed geodesic, and the shortest distance to the first conjugate points along geodesics. The estimate of  $i(M)$  in terms of geometric quantities plays an important role in the pinching problems and finiteness theorem of topological types. Let  $M$  be a compact simply connected Riemannian manifold whose sectional curvature  $K$  satisfies  $(0 <) \delta \leq K \leq 1$ . If  $\dim M$  is even then we have  $i(M) \geq \pi$  for any  $\delta > 0$  (Klingenberg, [K], [C-E]). On the other hand for the odd dimensional case, it is known that  $i(M) \geq \pi$  if  $\delta \geq 1/4$  (Klingenberg, [C-G], [K-S]). But for  $\delta < 1/9$  there are examples with  $i(M) < \pi$  due to Berger ([C-E]).

Now Burago-Toponogov [B-T] assert the following: Let  $M$  be a compact simply connected  $C^3$ -Riemannian manifold of dimension 3 which satisfies

$$(*) \quad K \leq 1, \quad \text{Ricci}(v) \geq R(>0) \quad \text{for a positive constant } R,$$

where  $\text{Ricci}(v)$  denotes the Ricci curvature of unit tangent vector  $v$ . Then  $i(M) \geq 6 \exp(-6/R)$ . (Note that  $R \leq 2$ .)

Although their proof is very interesting, there seems to be a serious gap. In the present note we fill this gap. Moreover we improve the estimate in the following form:

**Theorem.** *Let  $M$  be a compact connected  $C^3$ -Riemannian manifold of dimension 3 which satisfies (\*). Then for any  $b > 1$  we have the following estimate*

$$(1.1) \quad i(M) \geq \min \left\{ \frac{2b(b-1)\pi^2}{b^2\pi^2 + (b-1)^2}, \pi \{1 + (1/\pi^2) \cdot (e^{b/R} - 1)^4\}^{-1/2} \right\}.$$

**Remark 1.** This is still far from the best possible result even in the case of constant curvature ( $K = 1, R = 2$ ). (1.1) also means that for given  $b > 1$  with