

# On Power Compactness in a Banach Space

JOHN J. BUONI, ALBERT KLEIN,  
BRIAN M. SCOTT & BHUSHAN L. WADHWA

**1. Introduction.** Let  $X$  and  $Y$  be complex infinite dimensional Banach spaces, and let  $B(X,Y)$  denote the collection of all bounded linear operators from  $X$  to  $Y$ . We shall say that a sequence  $(T_n)$  in  $B(X,Y)$  has property P if for every bounded  $(x_n)$  in  $X$  there exists a positive integer  $k$  such that the sequence  $(T_k x_n)_{n=1}^\infty$  has a convergent subsequence. Similarly, if for each bounded  $(x_n)$  in  $X$  there is a positive integer  $k$  such that  $(T_k x_n)_{n=1}^\infty$  has a weakly convergent subsequence, we say that  $(T_n)$  has property Q. D. G. Tacon, using techniques from non-standard analysis, showed in [7] that every sequence of operators with property P (Q) contains a compact (weakly compact) operator. For compact operators on a Hilbert space the result has also been obtained by Jose Barria [1]; his proof is based on a deep result of D. Voiculescu [8]. The purpose of this note is to prove both results in the Banach space setting, using results involving the representation of the Calkin algebra and generalized Calkin algebra [2], [3], [4]. However, the authors would like to point out that many of these ideas are direct translations to a standard setting of Tacon's elegant results.

The remainder of the paper is divided into three parts. We first use Chadwick and Wickstead's construction in [4] of a representation of the Calkin algebra to show that any sequence  $(T_n)$  in  $B(X,Y)$  satisfying property P contains a compact operator. This part is quite straightforward, since the idea of the proof is not lost in a forest of technicalities. Then we generalize the result of Chadwick and Wickstead to obtain a representation of the generalized Calkin algebra,  $B(X,X)/W(X,X)$ , where  $W(X,Y)$  is the collection of all weakly compact operators from  $X$  to  $Y$ . Finally, we use this new construction to show that any sequence  $(T_n)$  in  $B(X,Y)$  satisfying property Q contains a weakly compact operator.

To this end we establish some notation. If  $\mathcal{U}$  is a free ultrafilter of  $N$ ,  $x \in \ell_\infty(X)$ , and  $y \in X$ , we say that  $x$  converges to  $y$  along  $\mathcal{U}$ , written

$$\begin{array}{c} \mathcal{U} \\ x \rightarrow y, \end{array}$$

iff for each  $\varepsilon > 0$  the set

$$\{i \in N : \|x_i - y\| < \varepsilon\}$$

belongs to  $\mathcal{U}$ . Similarly,  $x$  weakly converges to  $y$  along  $\mathcal{U}$ , written