Nonstable Ergodic Homeomorphisms of R^4

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1. Introduction. V. Prasad [9] has recently demonstrated the existence of ergodic transformations in the spaces $\mathcal{H}(R^n)$, $n \geq 2$, consisting of all homeomorphisms of R^n onto itself which preserve Lebesgue measure m. He did this by applying the Baire Category Theorem to show that the ergodic transformations are generic (dense G_δ) in the spaces $\mathcal{H}(R^n)$, $n \neq 4$, and also in the closed subset of $\mathcal{H}(R^4)$ consisting of *stable* homeomorphisms. (A homeomorphism is called stable if it is the finite composition of homeomorphisms which are equal to the identity on nonempty open sets.) The restriction to $n \neq 4$ (or to stable homeomorphisms) arises in Prasad's work through his use of the Annulus Conjecture (or Theorem for $n \neq 4$—see Kirby [6]). Thus Prasad's paper leaves open the possibility that ergodicity is not generic in $\mathcal{H}(R^4)$.

In this paper we show that ergodicity *is* generic in $\mathcal{H}(R^4)$. In particular we establish the existence of an ergodic transformation in $\mathcal{H}(R^4)$ which is not orientation preserving and the conditional existence of an ergodic, nonstable, orientation-preserving transformation in $\mathcal{H}(R^4)$ given the existence of a nonergodic one. The Annulus Conjecture is avoided by performing some of the constructions within the larger space of all measure-preserving bijections of R^4.

The above discussion, as well as the title of the paper, emphasises dimension four. This is because the main results of this paper are new for that dimension only. However the techniques presented here are more general, covering equally well all dimensions $n \geq 2$, as well as all properties (such as recurrence, weak mixing and infinite ergodic index) which are generic in the purely measure theoretic context. The extension of Prasad's work to these properties was already accomplished in a previous paper [2], but that paper assumed Prasad's results and consequently maintained his assumption that $n \neq 4$. In Section 2 we prove (Theorem 5), for all $n \geq 2$, that any conjugate-invariant property which is generic in the space $\mathcal{M}(R^n)$ (consisting of all invertible m-preserving transformations of R^n) with respect to the weak (or coarse) topology is also generic in $\mathcal{H}(R^n)$ with respect to the compact-open topology. For the convenience of those wishing to compare Section 2 with [2], we have numbered all the propositions to correspond with those of [2]. The main differences are in Theorem 1 which is weaker than that of [2], and Theorem 3 which is stronger.

The results of Section 2 do not constitute a complete extension to noncompact manifolds of the work of Oxtoby and Ulam [8] on compact manifolds, because

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