

# *Existence Theory and Qualitative Properties of the Solutions of Some First Order Quasilinear Variational Inequalities*

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## Introduction

This paper deals with the existence, the uniqueness and the qualitative properties of the solutions of the Cauchy problem

$$(C.P.)_f \quad \frac{\partial u}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} (\Phi_i(u)) + \beta(u) \ni f \quad \text{in } (0, +\infty) \times \mathbf{R}^N,$$

$$u(0, \cdot) = u_0(\cdot) \quad \text{in } \mathbf{R}^N,$$

as well as those of the stationary equation

$$(S.P.)_f \quad \sum_{i=1}^N \frac{\partial}{\partial x_i} (\Phi_i(u)) + \beta(u) \ni f \quad \text{in } \mathbf{R}^N,$$

where  $\Phi = (\Phi_1, \dots, \Phi_N)$  is a continuous function from  $\mathbf{R}$  into  $\mathbf{R}^N$  and  $\beta$  a maximal monotone graph of  $\mathbf{R}^2$  [6].

When  $\beta$  is a smooth function, the equation  $(C.P.)_0$  is known as a balance law equation (a conservation law if  $\beta = 0$ ). It has been studied by numerous authors, namely: Burger, Hopf, Lax, Oleinik, Vol'pert, Kruzkov and it is well known that even if all the data are smooth, every solution of  $(C.P.)_0$  becomes in general discontinuous after a finite time [22]. Thus the equation must be taken in a weak sense. However, there may exist an infinite number of weak solutions of  $(C.P.)_0$  and an additional principle called the entropy condition is needed to select the unique physical weak solution. The form of the entropy condition which will be used here was first given by Vol'pert [29] and then improved by Kruzkov in his penetrating work [21].

As for the hypothesis on  $\beta$ , our assumption of monotonicity allows us to eliminate the assumption of Lipschitz continuity made in previous works, [2], [11] and [23]. We can even consider the case when  $\beta$  is a multivalued function, for example the evolution variational inequality of the type