

Bundles of Low Codimension over \mathbf{CP}^n

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We consider complex n -plane bundles ξ over n -dimensional complex projective space \mathbf{CP}^n . Our object is to prove that if $c_{n-s+1}(\xi) = \dots = c_n(\xi) = 0$, then ξ has s cross-sections which are linearly independent over the complex numbers provided $s \leq \rho(n-s)$. Here $\rho(k)$ is an appropriate function for which we do not claim that it is best possible but merely that it is better than the previously published results in this direction.

Since n -bundles over \mathbf{CP}^n are stable and since integers c_1, \dots, c_n are the Chern classes of a stable bundle $\xi \downarrow \mathbf{CP}^n$ if and only if they satisfy a certain arithmetic condition S_n (cf. [2]), we may formulate our result as follows.

Theorem. *Let c_1, \dots, c_r be r integers satisfying the Riemann-Roch-Schwarzenberger condition S_n . Then there is an r -dimensional complex bundle $\zeta \downarrow \mathbf{CP}^n$ with Chern classes $c_i(\zeta) = c_i$, $1 \leq i \leq r$, provided $r \geq 3$ and $n \leq r + \rho(r)$, where*

$$\rho(r) = \begin{cases} 3 & \text{if } r \equiv 0, 1 \pmod{4} \\ 4 & \text{if } r \equiv 2, 3 \pmod{4}. \end{cases}$$

Remarks. 1). The case $r = 1$ is of course special: for any integer c_1 there is a line bundle $\lambda \downarrow \mathbf{CP}^n$ with $c_1\lambda = c_1$. For 2-bundles we have shown in [1] that there is a nontrivial obstruction to the existence of a 2-bundle $\zeta \downarrow \mathbf{CP}^5$ with given Chern classes c_1, c_2 satisfying S_5 . The obstruction vanishes if c_1 is odd and for c_1 even it is equal to

$$\frac{\Delta^2(\Delta - 1)}{24} \in \mathbf{Q/Z}, \quad \Delta = \frac{c_1^2 - 4c_2}{4}.$$

2). From [3] it follows that $\rho(r) \geq 2$ if r is odd.

3). By pushing the techniques used to prove the theorem above a bit further one can even prove for $r \geq 4$

$$\rho(r) = \begin{cases} 4 & \text{if } r \equiv 0 \pmod{4} \text{ unless } c_1 \text{ is even, } c_2 \text{ is odd} \\ & \text{or if } r \equiv 1 \pmod{4} \text{ unless } c_2 \text{ is odd} \\ 5 & \text{if } r \equiv 2 \pmod{4} \text{ unless } c_1 \text{ is even or} \\ & \text{if } r \equiv 3 \pmod{4} \text{ unless } c_1^2 + c_2 \text{ is odd.} \end{cases}$$