

Regularity for Elasto-Plastic Type Variational Inequalities

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1. Introduction. We develop $W^{2,p}$ regularity results for a class of second order elliptic variational inequalities. In order to describe the nature of this class it is convenient to consider only those variational inequalities which arise as minimization problems. The solution of such a variational inequality is the function which minimizes a convex functional, J , over some closed convex set of functions, \mathbf{K} . The class of variational inequalities for which we derive regularity results is determined by the form we require \mathbf{K} to have.

Our results are on second order elliptic variational inequalities and the canonical example of a functional of this type is

$$J(w) = \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^n (D_i w)^2(x) - (fw)(x) \right\} dx,$$

where Ω is a domain of \mathbf{R}^n , $D_i = \partial/\partial x_i$, $f \in L^2(\Omega)$ and J is defined on $W^{1,2}(\Omega)$. For this example our results show that if $f \in L^p(\Omega)$ and

$$(1.0) \quad \mathbf{K} = \{w \in W^{1,2}(\Omega) : F(x, w(x), Dw(x)) \leq 1$$

for a.e. x in Ω and $w|_{\partial\Omega} = g\}$,

where $F \in C^2(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$ is convex in the last two variables (plus some technical assumptions), $D = (D_1, D_2, \dots, D_n)$ and g is some continuous function on $\partial\Omega$, then the solution of the variational inequality associated to J and \mathbf{K} is in $W_{loc}^{2,p}(\Omega)$. Furthermore, if $\partial\Omega$ is smooth near a point $x_0 \in \partial\Omega$ and g is smooth near x_0 then the solution of the variational inequality is in $W^{2,p}(\Omega \cap B(x_0, r))$ for some positive number r , where $B(x_0, r)$ is the ball about x_0 of radius r .

We call the condition,

$$F(x, w(x), Dw(x)) \leq 1 \quad \text{for a.e. } x \text{ in } \Omega,$$

a generalized gradient constraint or g.g.c., for short. A convex set, \mathbf{K} , of the form given by (1.0) is said to be determined by a g.g.c. and a variational inequality associated to a convex set determined by a g.g.c. will be called a g.g.c. problem. Our regularity results are for g.g.c. problems.

Our results are proved by adapting an idea due to Brezis and Stampacchia [2]