

Bounded Orbits of Conjugation, Analytic Theory

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Introduction. Throughout the text, \mathcal{H} will signify a Hilbert space of generally unspecified dimension; separable will always be taken to mean countably infinite dimensional. $\mathcal{B}(\mathcal{H})$ will denote the collection of bounded linear operators on \mathcal{H} ; the term operator will connote bounded linear. \mathbb{C} will denote the complex field, and \mathbb{C}^* will denote $\mathbb{C} \setminus \{0\}$. \mathbb{R} will denote the reals, \mathbb{R}^+ the non-negative reals, \mathbb{Z} the integers and \mathbb{Z}^+ the non-negative integers. Also, we will frequently identify \mathbb{C} with the collection $\{\lambda I : \lambda \in \mathbb{C}\}$ where I is the identity operator on \mathcal{H} . That is, we will refer to λI as a scalar operator, or scalar, and write it more simply as λ . The context of this usage should preclude any confusion.

The following definition is due to Deddens [1].

Definition 0.1. Let A be invertible in $\mathcal{B}(\mathcal{H})$.

$$\mathcal{B}_A^D = \{X \in \mathcal{B}(\mathcal{H}) : A^n X A^{-n} \text{ is uniformly norm bounded for } n \in \mathbb{Z}^+\}.$$

Thus, \mathcal{B}_A is the collection of operators having bounded conjugation orbits, conjugation being by A . In the cited paper, an important connection is made:

Theorem 0.2. (Deddens). *Let \mathcal{H} be separable. Let A be positive and invertible in $\mathcal{B}(\mathcal{H})$. Let μ denote the spectral measure of A , and let \mathcal{N} denote the completion of the nest*

$$\{\mu[0, \varepsilon]\mathcal{H} : 0 \leq \varepsilon\}.$$

Let $\mathcal{A}_{\mathcal{N}}$ denote the nest algebra of \mathcal{N} ; that is, the algebra

$$\{X \in \mathcal{B}(\mathcal{H}) : X\mathcal{S} \subset \mathcal{S} \text{ for } \mathcal{S} \in \mathcal{N}\}.$$

Then $\mathcal{A}_{\mathcal{N}} = \mathcal{B}_A$. Moreover, if \mathcal{N} is a complete nest of subspaces of \mathcal{H} , then $\mathcal{B}_{\mathcal{N}} = \mathcal{B}_A$ for some positive, invertible element A in $\mathcal{B}(\mathcal{H})$.

Thus, \mathcal{B}_A was introduced as an alternative for the description of nest algebras; but the definition does have greater generality. If \mathcal{H} is finite dimensional, then \mathcal{B}_A is completely determinable for any A invertible in $\mathcal{B}(\mathcal{H})$ [6]. If \mathcal{H} is infinite dimensional, then no such statement can be made; however, one always has the following inclusions