

Newman Polynomials on $|z| = 1$

DOUGLAS M. CAMPBELL, HELAMAN R. P.
FERGUSON & RODNEY W. FORCADE

1. Introduction. Let $\lambda_1, \dots, \lambda_n$ be distinct real numbers. W. Hayman and S. Rudolfer asked [1, problem 7.33] about

$$\|\Lambda\| = \inf_{-\infty < t < \infty} \left| \sum_{j=1}^n e^{i\lambda_j t} \right|,$$

where $\Lambda = (\lambda_1, \dots, \lambda_n)$. An elementary argument of theirs shows $\|\Lambda\| \leq (n-1)^{1/2}$, a result adequate for the prediction theory application in which the problem had its origin. They indicated that they were unaware of any elements Λ for which $\|\Lambda\| > 1$. If the λ 's are distinct positive integers, then the problem reduces to a question about the minimum modulus on $|z| = 1$ of a polynomial whose coefficients are either 0 or 1. This led D. Newman to pose as a separate question the existence of polynomials

$$\sum_{j=1}^n z^{m_j} \quad \text{with} \quad \left| \sum_{j=1}^n z^{m_j} \right| > 1$$

on $|z| = 1$.

In this paper we prove that Hayman's and Rudolfer's general question about the upper bounds of the infimum always reduces to a specific question about polynomials whose coefficients are 0 or 1.

We prove that if $\lambda_1, \dots, \lambda_n$ are distinct real numbers, then there are distinct positive integers m_1, \dots, m_n such that

$$\inf_{-\infty < t < \infty} \left| \sum_{j=1}^n e^{i\lambda_j t} \right| \leq \min_{|z|=1} \left| \sum_{j=1}^n z^{m_j} \right|.$$

We then answer Newman's question in the affirmative for such polynomials.

2. Preliminaries. For any vector space X and subsets A and B of X let

$$A + B = \{a + b : a \in A, b \in B\}.$$

Distance and convergence on an m -dimensional vector space X , each of whose elements has unique representation $X = (x_1, \dots, x_m)$ will always be given by the norm $\|x\| = \sum_{j=1}^m |x_j|$.