

Convex Solutions to Nonlinear Elliptic and Parabolic Boundary Value Problems

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When are solutions to elliptic or parabolic boundary value problems given by convex functions? Extending a maximum principle derived in an earlier paper to a slightly larger class of elliptic equations and to related parabolic equations yields some new answers to this question as well as some new proofs of already known results.

In particular this paper contains:

- (a) A proof that a function on a convex domain Ω whose graph makes zero contact angle with the bounding cylinder $\partial\Omega \times \mathbf{R}$ and which satisfies an elliptic equation of the appropriate type is convex.
- (b) A generalization and direct proof of the Brascamp-Lieb result that the first eigenfunction of the Laplacian on a convex domain is Log concave (and so has convex level sets).

In order to get these results the nature of the maximum principle used requires strong constraints on the boundary behavior of u . The reason these constraints are necessary is illustrated with a simple counterexample.

The statements and proofs of the maximum principles are given in Section 1. They are used in Section 2 to prove the convexity results. The counterexample and some related unsolved problems are briefly discussed in Section 3.

I have learned that L. A. Caffarelli and J. Spruck have also (independently and concurrently) extended the results of the earlier paper [7]. Their work will apparently include much of what is presented here, as well as some further applications [2].

Section 1. Maximum principles. We introduce the concavity function and prove several maximum principles for it.

The concavity function \mathcal{C} was first introduced in [7]. It is a natural way to measure by how much a function u fails to be convex. Let u be defined on the closure of a bounded domain Ω . Let $0 \leq \mu \leq 1$. Then for

$$(1) \quad y_1, y_3 \in \bar{\Omega} \quad \text{such that} \quad y_2 = \mu y_3 + (1 - \mu)y_1 \in \bar{\Omega}$$

define

$$(2) \quad \mathcal{C}(y_1, y_3, \mu) = u(y_2) - \mu u(y_3) - (1 - \mu)u(y_1).$$