

*Maximum Principle for Parabolic
Inequalities and the Heat Flow
on Open Manifolds*

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In this paper we study the questions of existence and uniqueness for the Cauchy problem for the heat equation on a Riemannian manifold from the point of view of partial differential equations. The main uniqueness result is a consequence of a generalized maximum principle (due to Calabi [C] in the case of elliptic equations), and asserts that bounded solutions of the heat equation are determined uniquely by their initial values if the manifold is complete and has Ricci curvature bounded from below. The same method gives uniqueness for the heat equation on forms of arbitrary degree and yields a pointwise estimate leading to a new proof of a theorem of Yau [Y1] stating that bounded solutions which vanish at spatial infinity initially will vanish at infinity for all time. Another consequence of the uniqueness theorem of this paper is that the conservation law (cf. [G]) for the heat equation holds for complete manifolds with Ricci curvature bounded below.

Our uniqueness theorem should be compared with a result of Vauthier [V]. Using Brownian motion techniques he proves that solutions of the heat equation which are in certain weighted L^p spaces in the space variables are uniquely determined by their initial values. However, he has to require that the manifold be complete, that it have uniform bounds for the curvature tensor with its first two covariant derivatives, and have a lower bound for its injectivity radius. Our condition that the Ricci curvature be bounded below is much nicer from the geometric point of view. L. Karp [K] has recently proved a uniqueness theorem more general than ours. His proof, however, does not yield any pointwise estimates.

Regarding existence we give a proof that a unique smallest positive fundamental solution exists for an arbitrary Riemannian manifold and defines a semigroup on the space of bounded continuous functions. The result itself is quite old and the literature on the subject is vast. However, to our knowledge all the proofs which impose no restriction on the manifold use probabilistic techniques (cf. e.g. [McK]). Our proof is similar to that of [McK], in that we use an exhaustion of the manifold by compact subsets, but we use only standard facts of partial differential equations.