

Stochastic Variational Inequality for Reflected Diffusion

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1. Introduction. Let $(y^\varepsilon(t), t \geq 0)$ be a diffusion in \mathbf{R}^n , starting at $x \in \bar{\mathcal{O}}$ and with the drift coefficient $g(y^\varepsilon(t), v(t)) - (1/\varepsilon)\beta(y^\varepsilon)$ and the diffusion term $\sigma(y^\varepsilon(t), v(t))$. The set $\bar{\mathcal{O}}$ is a convex bounded domain in \mathbf{R}^n and the function β represents a penalization factor, which is exactly the gradient of the square of the distance to the open set \mathcal{O} . The functions g, σ are given and $v(\cdot)$ stands for the control which is an adapted process taking values in a closed convex set V in \mathbf{R}^m . We are interested in the behavior of the process $y^\varepsilon(t)$ as ε decreases to zero.

We show that $y^\varepsilon(t)$ converges uniformly in $v(\cdot)$ to the process $y(t)$, which is the normal reflected diffusion on $\bar{\mathcal{O}}$, given as the unique solution of the corresponding stochastic variational inequality. The convergence is established uniformly in $0 \leq t \leq T < \infty$ for all T , and in the L^p -norm of the probability space where the diffusion $y^\varepsilon(t)$ is defined. This uniform convergence enables us to prove that we also have convergence of the associated nonlinear semigroup.

When the boundary of the region \mathcal{O} is smooth, several authors have studied the reflected diffusion. We refer to the books of Bensoussan-Lions [3], Ikeda-Watanabe [9], Gihman-Skorokhod [10], McKean [17], and the works of Freidlin [7], Krylov [11], Nakao-Shiga [23], Sato-Ueno [25], Stroock-Varadhan [29], and Venttsel [32]. In Tanaka [31], the case of a general convex domain was treated.

In this paper, we present a different way to establish the existence and uniqueness of a normal reflected diffusion in convex regions with boundary not necessarily smooth. The process is regarded as the unique solution of a stochastic variational inequality in a strong sense. We obtain some a priori estimates and then the associated semigroup can be considered. We remark that some results in this direction are given in Lions-Menaldi-Sznitman [15] and Shalauimov [26].

2. Formulation of the problem. Let (Ω, \mathcal{F}, P) be a probability space, $(\mathcal{F}^t, t \geq 0)$ be a nondecreasing right continuous family of completed sub- σ -fields of \mathcal{F} , and $w(t)$ be a standard Wiener process in \mathbf{R}^n with respect to \mathcal{F}^t .

Suppose that \mathcal{O} is an open set of \mathbf{R}^n satisfying

$$(2.1) \quad \mathcal{O} \text{ is convex and bounded,}$$

and denote by $\beta(x)$ the gradient of the square of the distance to the set \mathcal{O} , i.e.