

The Essential Commutant of CSL Algebras

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Since the paper of Johnson and Parrot [9] characterized the commutant modulo the compact operators of certain von Neumann algebras, attention has been drawn to the essential commutant of other operator algebras. The problem was raised for Toeplitz operators by Douglas [5], and solved by the author in [3]. The essential commutant of a nest algebra was found in [2] by Christenson and Peligrad. Nest subalgebras of von Neumann algebras were studied by Gilfeather and Larson [6], and they extended some of these results to certain commutative subspace lattices of finite width [7]. In [4], the author studied essentially invariant projections for CSL algebras which gives some insight into essential commutant as well.

In this paper, we characterize the essential commutant of separably acting CSL algebras. Our proof depends on Arveson's representation theorem for CSL's [1], as projections onto essentially increasing sets in $L^2(X, \mu, \leq)$. The essential commutant is described in terms of "slowly oscillating" functions on X relative to the partial order \leq . If \mathcal{L} is atomic, the separability condition is unnecessary. The general, nonseparable case is reduced to the separable one, but the notion of "slowly oscillating" function is lost. Gilfeather and Larson have some similar results by different methods. Their results will appear independently [8].

Section 2 deals with the atomic case, in which the ideas are most transparent. Sections 3 and 4 generalize the results to arbitrary CSL lattices. In Section 5, we collect various applications and examples.

1. Preliminaries. \mathcal{H} will be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on \mathcal{H} , and $\mathcal{K}(\mathcal{H})$ the closed ideal of compact operators. The quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra will be denoted by π . All projections will be self adjoint. Let \mathcal{L} be a subspace lattice, meaning a collection of subspaces of \mathcal{H} closed under unions, intersections and closed in the strong operator topology. We will identify a subspace with the projection onto it. A lattice \mathcal{L} is *commutative* if all the projections in \mathcal{L} commute with one another. We consider only commutative subspace lattices in this paper. $\text{Alg } \mathcal{L}$ is the algebra of operators leaving \mathcal{L} invariant. By [1] and [4], $\text{Lat alg } \mathcal{L} = \mathcal{L}$. The core of $\text{alg } \mathcal{L}$ is \mathcal{L}'' .

If \mathfrak{A} is an algebra, the essential commutant of \mathfrak{A} , \mathfrak{A}^{ec} , is the set of operators T such that $TA - AT$ is compact for all A in \mathfrak{A} . Since \mathcal{L}' is contained in $\text{alg } \mathcal{L}$, $(\mathcal{L}')^{ec}$ contains $(\text{alg } \mathcal{L})^{ec}$. By [9], $(\mathcal{L}')^{ec}$ equals $\mathcal{L}'' + \mathcal{K}(\mathcal{H})$. As was noted in [7, Lemma 1.2], if T belongs to $(\text{alg } \mathcal{L})^{ec}$, then $\pi(T)$ is normal. So by Fuglede's