

Interpolation of Sobolev Spaces. The Real Method

CALIXTO P. CALDERÓN & MARIO MILMAN

1. Introduction. In [4] DeVore and Scherer obtained an explicit expression of the K functional for the couple of Sobolev spaces $(W_1^k(\mathbf{R}^n), W_\infty^k(\mathbf{R}^n))$, namely

$$(1.1) \quad K(f, t, W_1^k, W_\infty^k) \approx \sum_{|\alpha| \leq k} \int_0^t (D^\alpha f)^*(s) ds$$

where $*$ stands for non-increasing rearrangement and the symbol \approx denotes equivalence up to constants.

An immediate consequence of (1.1) is

$$(1.2) \quad (W_1^k(\mathbf{R}^n), W_\infty^k(\mathbf{R}^n))_{\theta, q} = W_{p_{\theta q}}^k(\mathbf{R}^n),$$

where $0 < \theta < 1$, $1/p_\theta = 1 - \theta$, $1 \leq q \leq \infty$.

These results can be extended in a simple way to Sobolev spaces on domains Ω with minimally smooth boundary in the sense of Stein [9] as indicated in [4].

The methods of [4] are rather involved and use results from the theory of splines and ad hoc covering lemmas.

In this paper we provide a simple proof of a more general version of (1.1) by using maximal operator techniques. Our results allow, for instance, the computation of the interpolation spaces

$$(W_{L^{p_0}(\text{Log } L)^{\alpha_0}}^k, W_{L^{p_1}(\text{Log } L)^{\alpha_1}}^k)_{\theta, q}.$$

2. Preliminaries. We consider rearrangement invariant spaces (r.i. spaces) $X(\mathbf{R}^n)$ of locally integrable functions on \mathbf{R}^n . It is well known and easy to prove (cf. [6]), that there exists an r.i. space $\hat{X}(0, \infty)$ such that

$$(2.1) \quad \|f\|_X = \|f^*\|_{\hat{X}}$$

where $f^*(t) = \inf\{s : |\{x : |f(x)| > s\}| \leq t\}$, is the non-increasing rearrangement of f .

The Sobolev spaces $W_X^k(\mathbf{R}^n)$, $k \in \mathbf{N}$, are defined in the usual fashion using the norms

$$(2.2) \quad \|f\|_{W_X^k} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_X.$$