

# *Hyperbolic Convexity and Level Sets of Analytic Functions*

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**1. Introduction.** Let  $D$  be a domain in  $\bar{\mathbb{C}}$  with at least three boundary points and let  $f$  be an analytic universal covering mapping of  $\Delta = \{z: |z| < 1\}$  onto  $D$ . A *level set* of  $f$  is a set  $\{z \in \Delta: f(z) \in L\}$  where  $L$  is a line or circle meeting  $D$ . Hayman and Wu have shown that each level set of a univalent  $f$  has length less than  $10^{37}$ ; using an alternative argument, Garnett, Gehring and Jones have obtained a smaller, but still relatively crude bound. See [3] and [4]. On the other hand, there are nonunivalent coverings with level sets of infinite length [1].

We study the geometry of special level sets, considering only the case in which one component,  $H$ , of  $\bar{\mathbb{C}} - L$  lies in  $D$ . Using a result of V. Jørgensen, we show in our two corollaries that each subdomain of  $\Delta$  mapped conformally onto  $H$  by  $f$  has perimeter bounded by  $\pi^2/2$  times its diameter. Thus each part of the level set corresponding to the boundary in  $\Delta$  of one of these subdomains has length no greater than  $\pi^2$ , and this bound is sharp. If  $D$  is simply connected, we see that the entire level set has length less than  $\pi^2$ , but the sharp bound in this case is not known.

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**2. The hyperbolic metric and hyperbolic convexity.** Let  $D$  be a simply or multiply connected domain in  $\bar{\mathbb{C}}$  with at least three boundary points: we call such  $D$  *hyperbolic domains*. If  $f: \Delta \rightarrow D$  is the analytic universal covering projection of  $\Delta$  onto  $D$ , then  $f$  has a well-defined local inverse in  $D$  and we define the hyperbolic distance in  $D$  between two points  $z_1, z_2 \in D$  to be

$$h_D(z_1, z_2) = \inf_{\gamma} \int_{\gamma} 2|(f^{-1})'(z)|(1 - |f^{-1}(z)|^2)^{-1}|dz|$$

where the infimum is taken over all curves  $\gamma$  in  $D$  from  $z_1$  to  $z_2$ . The hyperbolic distance is a conformal invariant, and if  $D$  is simply connected, there is a unique path in  $D$  from  $z_1$  to  $z_2$  having shortest hyperbolic length. When  $D$  is a disk, this shortest path is the arc  $\widehat{z_1 z_2}$  of the circle through  $z_1$  and  $z_2$  that is orthogonal to  $\partial D$ .