

Eigen-values and Eigen-vectors of Compact Composition Operators on $H^p(\Omega)$

STEPHEN D. FISHER

Let Ω be a domain (an open, connected set) in the complex plane with at least three boundary points. For $0 < p < \infty$ the space $H^p(\Omega)$ consists of those holomorphic functions f on Ω for which the subharmonic function $|f(z)|^p$ has a harmonic majorant on Ω ; $H^\infty(\Omega)$ is the space of bounded holomorphic functions on Ω . $H^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$ and $H^2(\Omega)$ is a Hilbert space; see [5]. Suppose that ϕ is a non-constant holomorphic function mapping Ω into Ω . If $f \in H^p(\Omega)$, then $f \circ \phi$ is also in $H^p(\Omega)$ because if u is a harmonic majorant of $|f|^p$, then $u \circ \phi$ is a harmonic majorant of $|f \circ \phi|^p$. Thus, the linear operator C_ϕ defined by

$$(1) \quad C_\phi f = f \circ \phi$$

maps $H^p(\Omega)$ into $H^p(\Omega)$, $0 < p \leq \infty$. It is the purpose of this note to draw some conclusions from the assumption that C_ϕ is a compact operator on some $H^p(\Omega)$, $1 \leq p < \infty$; the case when C_ϕ is compact on $H^\infty(\Omega)$ is explored thoroughly in [7]. The results contained here extend results in [1] and [2] which were derived for the case $p = 2$ and $\Omega = \{z : |z| < 1\}$. We assume throughout that $H^p(\Omega)$ is non-trivial.

If the function ϕ is a one-to-one conformal mapping of Ω onto Ω then C_ϕ maps $H^p(\Omega)$ onto $H^p(\Omega)$ and so C_ϕ cannot be compact. According to results in [4] this leaves only two possibilities:

- (2) there is a point $a \in \Omega$ with $\phi(a) = a$, $\phi'(a) = \mu$, $|\mu| < 1$, and $\phi_n(z) \rightarrow a$ for all $z \in \Omega$ as $n \rightarrow \infty$

or

- (3) there is a closed connected set C in $\partial\Omega$ such that C is the totality of limit points of $\{\phi_n(z)\}_{n=1}^\infty$ for all $z \in \Omega$.

Here ϕ_n is the n^{th} iterate of ϕ defined by

$$(4) \quad \phi_1(z) = \phi(z), \quad \phi_{n+1}(z) = \phi(\phi_n(z)), \quad n = 1, 2, \dots$$

If the situation in (2) occurs, then ϕ has an *attractive fixed point* at a . To state the first theorem, we need one more definition. A point $\lambda \in \partial\Omega$ is a *peak point*