

Regularity of the Derivatives of Solutions to Certain Degenerate Elliptic Equations

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Introduction. In this paper we show for fixed p , $1 < p < \infty$, that if u is a weak solution to $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ in a bounded domain $D \subseteq \mathbf{R}^n$, then ∇u is Hölder continuous on compact subsets of D . This result extends work of Evans [2] and Ural'tseva [11]. To be more specific we shall need some notation.

Let x be a point in n -dimensional Euclidean space \mathbf{R}^n , $|x|$ the Euclidean norm of x , dx Lebesgue measure on \mathbf{R}^n , $|E|$ the Lebesgue measure of a measurable set $E \subseteq \mathbf{R}^n$, and $L^p(E)$, $1 < p < \infty$, the usual Lebesgue space of functions f on E with norm denoted by $\|f\|_p$. Let $B(x, r) = \{y : |y - x| < r\}$, $r > 0$, $x \in \mathbf{R}^n$. Given a domain $D \subseteq \mathbf{R}^n$, let $W_1^p(D)$, $1 < p < \infty$, denote the Sobolev space whose functional elements u and their distributional partial derivatives u_{x_i} , $1 \leq i \leq n$, belong to $L^p(D)$ with norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^n \|u_{x_i}\|_p.$$

Let $C_0^\infty(D)$ be the space of infinitely differentiable functions with compact support in D . Let $\dot{W}_1^p(D)$ be the closure in $W_1^p(D)$ of $C_0^\infty(D)$.

For fixed $p > 1$, let $u \in W_1^p(D)$ and put $\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$. Suppose that

$$(1.1) \quad \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = 0,$$

whenever $\phi \in \dot{W}_1^p(D)$. Here $|\nabla u|^{p-2} \nabla u$ is defined to be zero at each x where $\nabla u = 0$. Note that if u has continuous second partials in D , then (1.1) and the divergence theorem imply

$$(1.2) \quad \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

in D . Therefore, we shall call u a weak solution to (1.2) in D . We remark that the smoothness of u in (1.1) cannot be deduced directly from theorems on weak solutions of elliptic equations of divergence type, since (1.2) is degenerate. However, Ural'tseva [11] and recently Evans [2] have shown that ∇u is Hölder continuous on compact subsets of D when $p \geq 2$. Moreover this result is sharp as