

# *Maximal Functions and Rearrangements: Some New Proofs*

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**1. Introduction.** We present an elementary treatment of the strong maximal function in  $R^n$  and its associated covering lemma, as well as a simple technique for obtaining analytic inequalities for other variants of the Hardy-Littlewood maximal function. While our treatment of the strong maximal function is suggested by the work of Córdoba and Fefferman [4], we have eliminated most of the geometry and functional analysis from their arguments. The result is an explicit, self-contained argument which can be used in other product spaces without the rich structure of  $R^n$ .

Although Hardy and Littlewood [7] used rearrangements in their original work with the maximal function, most authors, such as Stein [12], have preferred to work with distribution functions. Indeed, the standard arguments (using covering lemmas) lead immediately to an estimate for the distribution function of a maximal function. We show that the right sort of covering lemma also gives the rearranged maximal function immediately. The advantages of working with rearrangements are set forth clearly in the survey paper of Bennett and Sharpley [2]. We also give some new inequalities for rearrangements here.

One of the novel features of the work of Córdoba and Fefferman [4] is an exponential integrability estimate for certain sums of characteristic functions of rectangles, an estimate which is more or less dual to the basic inequality for the strong maximal function. Unfortunately, much of their proof was omitted and the estimate given is incorrect. We correct the estimate and give a direct proof using rearrangements, rather than the duality argument they suggest. As far as we can tell, the duality proof requires an additional step involving the invariance of  $R^n$  under dilations, whereas our argument is independent of geometry.

The heart of the work of Córdoba and Fefferman is a selection theorem for families of rectangles in  $R^n$ . Their arguments make extensive and ingenious use of the geometry of dyadic rectangles. However, we know of no reason why the dyadic case should suffice, and key portions of their argument fail in the non-dyadic case. In general, two rectangles with a common cross section can meet in a set whose measure is small compared to that of the original two rectangles; their method requires that this sort of behavior be ruled out. We employ a different selection process which treats the general case directly and simplifies the argu-