

# *The Partially Isometric Factor of a Semigroup*

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**1. Introduction.** Perhaps the best-known structure theorem for strongly continuous semigroups of operators on Hilbert space is a result of Cooper [1]: every isometric semigroup is unitarily equivalent to the direct sum of a unitary semigroup and a forward translation semigroup. This result was generalized in [3]: every partially isometric semigroup is unitarily equivalent to the direct sum of a unitary semigroup, a forward translation semigroup, a backward translation semigroup and a semigroup which decomposes into truncated shifts with respect to a direct integral decomposition.

A third structure theorem appears in [5]: every strongly continuous quasinormal semigroup  $(Q_t)$  is unitarily equivalent to the direct sum of a normal semigroup and a quasinormal weighted translation semigroup. ( $Q_t$  is quasinormal when  $Q_t$  commutes with  $Q_t^*Q_t$ .) The key to this decomposition is that  $(U_t)$  is a strongly continuous isometric semigroup where  $U_tP_t$  is the polar decomposition of  $Q_t$ . Consequently, Cooper's Theorem applies to  $(U_t)$ .

In this paper, we consider arbitrary strongly continuous semigroups  $(S_t)$  for which  $(U_t)$  is a semigroup and use the structure theorem for partially isometric semigroups to decompose  $(S_t)$  into the direct sum of four simpler semigroups (Theorem 4).

Three special semigroups play a crucial part in this paper. Two are defined on  $\mathcal{L}^2(\mathcal{H})$ , the Hilbert space of (equivalence classes of) weakly measurable functions from the nonnegative reals  $\mathcal{R}^+$  into a separable Hilbert space  $\mathcal{H}$ . The *forward translation semigroup*  $F = (F_t)$  is defined on  $\mathcal{L}^2(\mathcal{H})$  by

$$(F_t f)(x) = f(x - t) \quad \text{if } x \geq t \geq 0 \quad \text{and } 0 \quad \text{otherwise.}$$

The *backward translation semigroup*  $B = (B_t)$  is defined on  $\mathcal{L}^2(\mathcal{H})$  by

$$(B_t f)(x) = f(x + t) \quad \text{for all } x \geq 0 \text{ and } t \geq 0.$$

The *truncated translation semigroup of index*  $\alpha$  is defined by

$$(T_t f)(x) = f(x - t) \quad \text{if } 0 \leq t \leq x \leq \alpha \quad \text{and } 0 \quad \text{otherwise.}$$

The domain of  $(T_t)$  is the subspace  $\mathcal{L}^2(\mathcal{H}, \alpha)$  of  $\mathcal{L}^2(\mathcal{H})$  of all functions  $f$  with  $f(x) = 0$  for  $x > \alpha$ . It is understood that  $T_t = 0$  if  $t \geq \alpha$ .