

Some Min-Max Methods for the Hamilton-Jacobi Equation

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1. Introduction. In recent work Crandall and Lions [5], [6] have demonstrated a new uniqueness criterion for solutions of general nonlinear, first order PDE, and with this a new concept of weak solution.

The aim of this paper is to exploit their new notion of *viscosity solutions* (cf. [5]) to prove certain max-min representation formulas for the solution of the *Hamilton-Jacobi equation*

$$\begin{aligned}
 &u_t(x,t) + H(Du(x,t)) = 0 && 0 \leq t \leq T, x \in \mathbf{R}^n \\
 \text{(HJ)} \quad &\lim_{|x| \rightarrow \infty} u(x,t) = 0 && 0 \leq t \leq T \\
 &u(x,0) = u_0(x) && x \in \mathbf{R}^n.
 \end{aligned}$$

The main idea is to write $H(Du)$ as the max-min of certain affine functions of Du , to solve explicitly the analogous linear Hamilton-Jacobi equation for each of these affine functions, and then to construct the solution of (HJ) by repeated max-min operations on the solutions of the linear problems. The advantage of this is the relative simplicity of the resulting formula, (5.8), expressing the solution of the fully nonlinear problem (HJ) in terms of the (trivial) solutions of the linear equations.

This technique of converting a nonlinearity into max-min form is of course familiar from differential game theory (cf. Fleming [9], Friedman [10], [11]). Our construction is, however, more directly inspired by Nisio [14], who devised something similar for a wide class of problems with convex nonlinearities. The new trick here is an extension of her method (which entailed only repeated maxima) to cover (HJ) for nonconvex H . Nonlinear semigroup theory and in particular the Chernoff formula will provide appropriate theoretical justification.

As a further example, we verify in §6 that an explicit formula of E. Hopf [12] does indeed give the viscosity solution of (HJ) (if H is convex).

Finally, we refer the interested reader to Benton [2] for a nice account of various classical methods for studying (HJ). A good, more modern survey is P. L. Lions [13].