

# *Solutions of the Porous Medium Equation in $\mathbf{R}^N$ under Optimal Conditions on Initial Values*

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**Introduction.** This paper concerns the initial-value problem

$$(IVP) \quad \begin{aligned} u_t &= \Delta(|u|^{m-1}u) && \text{on } \mathbf{R}^N \times (0, T), \quad m > 1, \\ u(x, 0) &= u_0(x), && x \in \mathbf{R}^N, \end{aligned}$$

where the interval of existence  $[0, T)$ ,  $T > 0$ , depends on the initial data  $u_0$ . We are interested in solving (IVP) for the largest possible class of functions  $u_0$ . In fact, we will prove a nonlinear version of the result which states that the linear problem

$$(0.1) \quad \begin{aligned} u_t &= \Delta u && \text{on } \mathbf{R}^N \times (0, T), \\ u(x, 0) &= u_0(x), && x \in \mathbf{R}^N \end{aligned}$$

has a solution on some interval  $[0, T)$  if

$$(0.2) \quad \int |u_0(x)| e^{-c|x|^2} dx < \infty \quad \text{for some } c > 0.$$

(Of course, (0.2) guarantees that the solution formula for (0.1) provides a solution on  $[0, 4/c)$ .) If  $u_0 \geq 0$  it is known that (0.2) is also *necessary* for (0.1) to have a solution  $u \geq 0$  on some time interval  $[0, T)$ ,  $T > 0$  [15].

Here we prove, without the aid of explicit solutions, that (IVP) has a solution on some interval  $[0, T)$  if

$$(0.3) \quad \sup_{R \geq 1} R^{-(N+2/(m-1))} \int_{\{|x| \leq R\}} |u_0(x)| dx < \infty,$$

or, equivalently (see the Appendix),

$$(0.3)' \quad \sup_{R \geq 1} \frac{1}{R^N} \int_{\{|x| \leq R\}} \frac{|u_0(x)|}{(1 + |x|^2)^{1/(m-1)}} dx < \infty.$$