

Subharmonic Functions in $C^{1+\alpha}$ Domains

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In 1928, Littlewood [4] proved that if u is a subharmonic function in the unit disk with a nonnegative harmonic majorant, then u has radial limits almost everywhere on the unit circle. Many have since studied the extension of Littlewood's theorem to more general domains and to limits along more general curves. It is proved in [9] that if D is a starlike Lipschitz domain in \mathbf{R}^m , $m \geq 2$, and u is a subharmonic function in D with nonnegative harmonic majorant then u has radial limits almost everywhere with respect to the $(m-1)$ -dimensional measure on ∂D . (Starlikeness is imposed for the simplicity of the statement.) However, in [2], Dahlberg shows that for each α , $0 < \alpha < 1$, there is a $C^{1+\alpha}$ domain $D \subseteq \mathbf{R}^2$, a set F of positive length on ∂D and a negative subharmonic function u on D so that at every $P \in F$

$$\liminf u(Q) = -\infty \quad \text{as } Q \rightarrow P \text{ along } N(P),$$

where $N(P)$ is the inner normal of ∂D at P .

Every $C^{1+\alpha}$ domain is a Lipschitz domain, therefore the above-mentioned results from [2] and [9] are interesting by comparison. The reason behind Dahlberg's example is that the direction $N(P)$ is only $\text{Lip } \alpha$ with respect to P and the Riesz mass μ corresponding to u can be singular with respect to the Lebesgue measure.

When the domain is C^2 , the direction $N(P)$ is C^1 with respect to P . It follows from arguments in [2] or [9] that negative subharmonic functions on C^2 domains have limits along almost every $N(P)$.

In this paper, we let D be a $C^{1+\alpha}$ domain or a Dini domain in \mathbf{R}^m , $m \geq 2$, and u be a subharmonic function with nonnegative harmonic majorant in D . Assuming that the Riesz mass μ corresponding to u is absolutely continuous with respect to the Lebesgue measure, we shall give a condition, nearly best possible, on the density of μ , in terms of the distance to ∂D , to guarantee the almost everywhere existence of $\lim u(Q)$ as $Q \rightarrow P$ along $N(P)$ for $P \in \partial D$.

Let D be a domain in \mathbf{R}^m , $m \geq 2$, $G(P, Q)$ be the Green function on D , and μ be a nonnegative mass distribution on D . The integral

$$G\mu(P) = \int_D G(P, Q) d\mu(Q)$$

is called a Green potential given by μ if $G\mu$ is finite at some point in D . When