

On a Problem of H. Brezis and F. Browder Concerning Sobolev Spaces

JEAN-MICHEL CORON

I. Introduction. Let Ω be an open set of \mathbf{R}^N . We consider the real Sobolev space $W_0^{m,p}(\Omega)$, $m \geq 1$, $1 < p < +\infty$, that is the closure of $\mathcal{D}(\Omega)$ for the norm

$$\|u\|_{m,p} = \left(\int \sum_{|\alpha| \leq m} |D^\alpha u|^p dx \right)^{1/p}.$$

$W^{-m,p'}(\Omega)$ (with $1/p' + 1/p = 1$) is the dual of $W_0^{m,p}(\Omega)$. $\langle \cdot, \cdot \rangle$ is the scalar product in the duality between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$. $W^{m,p}(\Omega)$ is the set of functions of $L^p(\Omega)$ whose derivatives of order $\leq m$ are in $L^p(\Omega)$. Let $\mathcal{D}^+(\Omega)$ be the set of nonnegative functions of $\mathcal{D}(\Omega)$.

Let u be in $W^{m,p}(\Omega)$ and T be in $W^{-m,p'}(\Omega) \cap L_{\text{loc}}^1(\Omega)$. Let h be in $L_{\text{loc}}^1(\Omega)$. The purpose of this paper is to prove the following theorem.

Theorem 1. *If for every function φ in $\mathcal{D}^+(\Omega)$ we have*

$$(1) \quad \langle T, u\varphi \rangle \geq \int_{\Omega} h\varphi dx,$$

then

$$(2) \quad T(x)u(x) \geq h(x) \quad \text{a.e. } x \in \Omega.$$

Remarks. 1. This result answers a question raised in H. Brezis and F. Browder [1].

2. If, moreover, $h \in L^1(\Omega)$ it follows from (2) and a theorem of H. Brezis and F. Browder [2] that:

$$Tu \in L^1(\Omega) \quad \text{and} \quad \langle T, u \rangle = \int_{\Omega} T u dx.$$

3. If T is not in $L_{\text{loc}}^1(\Omega)$ but is a Borel measure finite on compact sets of Ω we write

$$T = t dx + \mu$$

where $t \in L_{\text{loc}}^1(\Omega)$, μ is a Borel measure finite on compact sets of Ω , dx and μ