

Inner Derivations with Closed Range in the Calkin Algebra

LAWRENCE A. FIALKOW
& DOMINGO A. HERRERO

To the memory of our friend and colleague James P. Williams

Let T be a bounded linear operator acting on an infinite dimensional Hilbert space \mathcal{H} and let $t = \pi(T)$ be its image in the Calkin algebra $\mathcal{A}(\mathcal{H})$. The inner derivation induced by t in $\mathcal{A}(\mathcal{H})$ has closed range if and only if the inner derivation induced by $T + K$ (for a suitable compact operator K) has closed range in the algebra $\mathcal{L}(\mathcal{H})$ of all operators acting on \mathcal{H} . Several other equivalences are obtained, including the (very simple) algebraic structures of T and t . The results also include an analysis of the similarity invariants of polynomially compact operators.

1. Introduction. In [1], C. Apostol characterized the Hilbert space operators which induce inner derivations having closed range. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space \mathcal{H} . An operator T in $\mathcal{L}(\mathcal{H})$ induces an inner derivation $\delta_T: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ defined by $\delta_T(X) = TX - XT$. Apostol's results (summarized below in Theorem 1.1) give necessary and sufficient conditions on an operator T so that $\text{Ran}(\delta_T)$, the range of δ_T , is norm closed in $\mathcal{L}(\mathcal{H})$.

It will be shown here that Apostol's results have surprisingly direct analogues in the case where $\mathcal{L}(\mathcal{H})$ is replaced by the Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators acting on \mathcal{H} . If $A, T, X, \dots \in \mathcal{L}(\mathcal{H})$ and $\pi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{A}(\mathcal{H})$ is the canonical quotient projection, then $\pi(A), \pi(T), \pi(X), \dots$ will be denoted by a, t, x, \dots . With this notation, our main result is as follows: the inner derivation δ_t induced by $t = \pi(T)$ on $\mathcal{A}(\mathcal{H})$ has closed range if and only if δ_{T+K} has closed range in $\mathcal{L}(\mathcal{H})$ for some compact perturbation $T + K$ of T . This result admits several alternate formulations in terms of the algebraic structures of T and t . To be more precise, we shall need some additional notation.

Recall that $T \in \mathcal{L}(\mathcal{H})$ is an *algebraic operator* (a *polynomially compact operator* respectively) if $p(T) = 0$ ($p(t) = 0$ respectively) for some nonconstant monic polynomial $p(z) = \prod_{i=1}^n (z - \lambda_i)^{m_i}$. In what follows p will always have this