

On Fixed Point Free Maps of the Complex Flag Manifold

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§1. Introduction. Let $T^n \subset U_n$ be a maximal torus. The homogeneous space

$$F(n) = \frac{U_n}{T^n}$$

is a complex manifold of complex dimension $n(n-1)/2$. Now $F(n)$ can be regarded as the space of orthogonal decompositions of \mathbb{C}^n into lines, so permutation of lines gives a free action of the symmetric group Σ_n on $F(n)$. Hence $F(n)$ admits a set of $n! - 1$ fixed-point-free maps, no two of which have a coincidence. In this paper we show that any set of $n!$ or more fixed-point-free maps on $F(n)$ contains a pair with a coincidence: in the language of §4, we show that $F(n)$ has noncoincidence index $n!$. The proof uses the Lefschetz fixed point and coincidence theorems, together with a classification of the endomorphisms of $H^*(F(n); \mathbb{Z})$.

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§2. The cohomology ring. As above, we regard $F(n)$ as the set of orthogonal decompositions of \mathbb{C}^n into lines. We evidently have n projection maps

$$\pi_i: F(n) \rightarrow \mathbb{C}P^{n-1}$$

given by taking the i^{th} line. Let $t \in H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$ be the usual generator, and let $t_i = \pi_i^* t \in H^2(F(n); \mathbb{Z})$. Then the following result is well known (see [1]).

Theorem 2.1. *The cohomology ring of $F(n)$ is given by*

$$H^*(F(n); \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_n] / \left(e_i(t_1, \dots, t_n) = 0, \ 1 \leq i \leq n \right)$$

where $e_i(t_1, \dots, t_n)$ is the i^{th} elementary symmetric function in t_1, \dots, t_n .

Remark. The free action of Σ_n on $F(n)$ mentioned in §1 evidently induces the action on $H^*(F(n); \mathbb{Z})$ which permutes the t_i .

Since t_i is pulled back from $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$, $t_i^n = 0$ for $1 \leq i \leq n$. In fact the only two-dimensional cohomology classes with this property are multiples of the