

Triangularizing Semigroups of Compact Operators

ERIC NORDGREN, HEYDAR RADJAVI
& PETER ROSENTHAL

There are several known theorems giving sufficient conditions that a multiplicative semigroup of finite matrices can be simultaneously triangularized. Kaplansky's Theorem ([5], page 137) states that a semigroup of finite matrices is simultaneously triangularizable if all the matrices in the semigroup have singleton spectra; this includes the nilpotent case (Levitzki's Theorem [9]) and the unipotent case (Kolchin's Theorem [7]). Other finite-dimensional results can be found in Kaplansky's paper [6]. We are grateful to Steve Schanuel for referring us to [6], which inspired us to consider Hilbert space analogs.

Our main results concern semigroups of quasinilpotent operators. We prove that a semigroup of quasinilpotent operators is simultaneously triangularizable if it is generated by a commutative set of compact operators together with some operators in Schatten p classes, (Corollary 2). This has the following corollary: if A is in some \mathcal{C}_p class and B is compact, and if all words in A and B are quasinilpotent, then $A + B$ is quasinilpotent, (Corollary 3). It seems that there should be a more direct proof of this corollary, but we have been unable to find one. We also point out another infinite-dimensional case of Kaplansky's Theorem and indicate related unsolved problems.

There have been previous papers on triangularization of collections of operators on Hilbert and Banach spaces: for Lie algebras of compact operators see [17] and [18], and for associative algebras of compact operators see [8].

The following definition is given in [8].

Definition. The collection \mathcal{S} of bounded linear operators is *simultaneously triangularizable* (abbreviated S.T.) if there is a maximal subspace chain each of whose members is invariant under all the operators in \mathcal{S} . (A maximal subspace chain is a chain of subspaces that is not properly contained in any other chain of subspaces.)

We begin with a theorem on simultaneous invariant subspaces. Definitions and properties of the \mathcal{C}_p classes can be found in [1], [3] and [16]; we assume $p \geq 1$.