

An Application of the Generalized Morse Index to Travelling Wave Solutions of a Competitive Reaction-Diffusion Model

C. CONLEY & R. GARDNER

§1. Introduction

This paper is concerned with the existence of travelling wave solutions of a diffusion-reaction system via the generalized Morse index of isolated invariant sets. The equations can be interpreted as a model for interspecific competition in which the two species diffuse over a “one-dimensional” spatial domain. The index theory is analogous to degree theory, and the method of proof follows lines familiar from the latter theory. The equations in question are “deformed” to a “standard” system where the index can easily be computed, and the existence theorem follows from the “non-triviality” of the index.

The index theory has been described in other papers; for completeness, an outline of the relevant details is included in §2. The “standard” system is used as an illustrative example throughout this discussion. The constructions are highly intuitive and they can be understood (and applied!) with only a cursory knowledge of the details of the index-theoretic machinery.

The main job here is to construct “isolating neighborhoods” which are analogous (in degree theory) to open sets with no critical points on the boundary. Some novel means of locating such neighborhoods are described.

The main theorem concerns a case where the reaction system is a competitive system: that is, the growth rate of one population decreases as the other population increases. In a plausible class of such models, each system admits three stable equilibria: the two at which one population is at a maximal stable level and the other is eliminated, and one at which *both* populations are at the zero level. In general there are (two) travelling waves connecting one or the other of the first two equilibria to the third. The theorem gives a criterion, in terms of the relative velocity of these two waves, in order that there be a third wave running between the first two equilibria.

A. The equations. Without diffusion, the model takes the form

$$(1) \quad \begin{aligned} u_1' &= u_1 F_1(u_1, u_2) \\ u_2' &= u_2 F_2(u_1, u_2) \end{aligned}$$