

Multipliers of Besov and Power-Weighted L^2 Spaces

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§0 Introduction. In [6] B. Muckenhoupt, R. L. Wheeden and W.-S. Young characterize the one-dimensional Fourier multipliers on the power-weighted L^2 space $L^2(|x|^{2\alpha})$ for all real α . (See also [7].) For example, they show that in the case where $\alpha - 1/2$ is positive and not an integer, the inequality

$$(0.1) \quad \int_{-\infty}^{\infty} |T_m f(x)|^2 |x|^{2\alpha} dx \leq C \int_{-\infty}^{\infty} |f(x)|^2 |x|^{2\alpha} dx$$

holds for all f in S_{00} if and only if m satisfies the Hörmander condition of order α . Here S_{00} denotes the space of infinitely differentiable rapidly decreasing functions on $(-\infty, \infty)$ whose Fourier transforms have compact support not including the origin. The multiplier operator T_m is defined by $(T_m f)^\wedge = m \hat{f}$ for f in S_{00} . For α a positive integer, the Hörmander condition of order α on m is

$$\|m\|_\infty^2 + \sup_{R>0} R^{2\alpha-n} \int_{R \leq |x| \leq 2R} |D^\alpha m(x)|^2 dx < \infty.$$

In addition, inequality (0.1) was characterized in the case $0 < \alpha < 1/2$ in terms of L^2 capacities of compact sets using a characterization of the trace inequality for Riesz potentials due to B. Dahlberg [4]. However the resulting condition has the defect that it is not always easy to determine whether a function m satisfies the condition.

In this paper we give extensions of these results to n -dimensional Euclidean space (for extensions in a slightly different direction see K. Andersen and W.-S. Young [2]). Our main theorem is that for $\alpha > n/2$ (with certain exceptions) the n -dimensional analogue of (0.1) holds if and only if m satisfies the n -dimensional Hörmander condition of order α . On the other hand, for $0 < \alpha < n/2$, $0 < \alpha \leq 1$ we give a new characterization of (0.1) and its n -dimensional analogue that does not involve L^2 capacities (we leave open the case $1 < \alpha < n/2$ as well as the exceptional cases where $\alpha - n/2$ or $(\alpha - n)/2$ is a non-negative integer). Our characterization is based on a new characterization of the L^2 trace inequality for Riesz potentials due to Ron Kerman and the author.

Finally we point out that by Plancherel's theorem, results on Fourier multipliers