A New Proof of the Closure Theorem for Integral Currents

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§0. Introduction

In a landmark paper of 1960, Federer and Fleming [FF] introduced a series of new techniques in the calculus of variations. Their approach modeled m-dimensional geometric objects in $\mathbf{R}^n$ as currents (i.e. continuous linear functionals on differential forms), and in the years since 1960 this approach has proven highly successful. Perhaps the most pivotal theorem in [FF] is one due to Fleming, which leads directly to the existence of geometrically acceptable solutions for a wide class of variational problems. This result is known as the *Closure Theorem,* a name expressing the following assertion: certain useful classes of *integral* currents (currents which, together with their boundaries, correspond to integration over rectifiable sets against integer density functions), form a weakly closed subspace of the much larger space of *normal* currents. The closure theorem allows one to solve many variational problems in a weak, operator setting, while still obtaining solutions which correspond to rectifiable geometric objects. Typically this is accomplished by simply passing to the limit of an extremizing sequence of integral currents.

One is tempted, in light of the above, to argue that the closure theorem is the central result of geometric measure theory. Yet, despite its fundamental importance, the original proof in [FF] has heretofore remained essentially the only proof, (see also [FH, 4.2.16]), though it is worth noting that, in codimension one, there is a proof of a related compactness theorem for sets of finite perimeter, due to DeGiori. (The latter is beautifully described in [G].)

Here we present a new proof of the closure theorem. Our main goal in seeking an alternative to the existing argument has been to liberate this theorem from its dependence on the highly technical, measure theoretic structure theory of H. Federer. While the latter theory elegantly characterizes the rectifiable subsets of a Euclidean space in terms of their projection properties, its proof is quite difficult, and has long been a significant obstacle to those seeking a full understanding of the closure theorem. The desirability of removing this obstacle is further enhanced by the fact that, since its role in proving the closure theorem, structure theory has seen few applications in the calculus of variations.

The proof we give here succeeds in circumventing structure theory, and in a

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