

# *Asymptotic Behavior, as $t \rightarrow +\infty$ of Solutions of Navier-Stokes Equations and Nonlinear Spectral Manifolds*

C. FOIAS & J. C. SAUT

*Dedicated to the 70th anniversary of Professor R. S. Phillips*

**0. Introduction.** It is well known that solutions of Navier-Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u + \nabla p \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0 \end{cases}$$

in  $\Omega \times ]0, \infty[$  ( $\Omega$  a smooth, open, bounded set in  $\mathbf{R}^n$ ,  $n = 2, 3$ ), with natural boundary conditions, are regular if  $n = 2$ , and become regular after a finite time (depending on the solution) if  $n = 3$ . Consequently, their energy

$$\int_{\Omega} |u(x, t)|^2 dx$$

and their enstrophy

$$\int_{\Omega} |\nabla \Lambda u(x, t)|^2 dx$$

decay at least exponentially in time, as  $t \rightarrow \infty$ .

This is essentially due to Leray [10], [11]. The aim of this paper is to show that this decay is exactly of exponential type, and is characterized by an eigenvalue of the Stokes operator. More precisely we show that the ratio of the enstrophy over the energy has a limit as  $t \rightarrow \infty$  which is an eigenvalue of the Stokes operator. This fact has direct consequences on the global structure of the Navier-Stokes equations; for instance, we give the first steps of an asymptotic expansion of  $u(t)$  on  $t \rightarrow \infty$ . This expansion will be completed in a forthcoming paper (see [6]), and used to construct a functional transform which, in some sense, linearizes the Navier-Stokes equations. Finally, we construct, in the space of initial data, nonlinear spectral manifolds, which determine completely the asymptotic behavior