

Circular Operators

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Introduction. A familiar theorem, due essentially to Weyl and von Neumann, asserts that every normal operator is a small compact perturbation of a diagonal operator. In particular, a normal operator with continuous spectrum can be compactly perturbed so as to have discrete spectrum.

In this paper we are concerned with operators which are highly nonnormal, but which in some sense have “continuous” or “discrete” features. Specifically, we show that no weighted translation operator is unitarily equivalent to a weighted shift; but that every weighted translation operator is unitarily equivalent to a small compact perturbation of a weighted shift. The second result requires Voiculescu’s theorem characterizing approximately equivalent representations of C^* -algebras [16], [4]. Some of the results below were announced in [8].

All Hilbert spaces considered in this paper are separable. $\mathcal{L}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ (or, more briefly, \mathcal{K}) will denote the bounded operators, the unitary operators, and the compact operators on the Hilbert space \mathcal{H} . The inner product on \mathcal{H} is denoted $\langle \xi, \eta \rangle$. If S is a set of operators, $C^*(S)$ denotes the C^* -algebra generated by S and the identity. If R is a set of vectors, $[R]$ denotes its norm closed linear span. Finally, T represents the compact abelian group of all complex numbers of modulus one and m is normalized Lebesgue measure on T .

The authors want to thank the referee for pointing out the paper of Gellar [7] which overlaps somewhat with the material in Section 1.

1. Circular operators and weighted shifts. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *weighted bilateral shift* if there is an orthonormal basis $\{e_n : n \in \mathbf{Z}\}$ for \mathcal{H} and a sequence w_n of complex numbers such that $Te_n = w_n e_{n+1}$. Note that some (or even all) of the weights may be zero, and necessarily we have $\sup_n |w_n| = \|T\| < +\infty$.

We introduce an invariant for single operators that will allow us to identify operators that are *not* weighted shifts. Fix $T \in \mathcal{L}(\mathcal{H})$; we denote by $G(T)$ the set of all complex numbers λ such that λT is unitarily equivalent to T .

Lemma 1.1. *If $T \neq 0$, then $G(T)$ is a subgroup of T .*

Proof. Let $\lambda \in G(T)$, then $|\lambda| \|T\| = \|\lambda T\| = \|U_\lambda T U_\lambda^{-1}\| = \|T\|$ and hence $|\lambda| = 1$. It is clear that $G(T)$ is a group. \square