

Representations of Operators between Function Spaces

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1. Introduction. Our objective in this paper is to study conditions on a symmetric quasi-Banach function space X on $[0,1]$ (see Section 2 for the precise definition) so that every (continuous) linear operator $T: X \rightarrow L_0[0,1]$ can be represented in the form

$$(1.1) \quad Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \quad \text{a.e.}$$

for $f \in X$, where the functions $a_n: [0,1] \rightarrow \mathbf{R}$ and $\sigma_n: [0,1] \rightarrow [0,1]$ are Borel and the series converges absolutely almost everywhere.

In [13] Kwapien obtained a representation of the form (1.1) for operators $T: L_0 \rightarrow L_0$, while in [11] we showed that there is such a representation for operators $T: L_p \rightarrow L_0$ where $0 < p < 1$ (see also [10] for operators $T: L_p \rightarrow L_p$). On the other hand it is clear that no finite-rank operator can be represented in the form of (1.1) so that in general we consider function spaces X with trivial dual.

Operators $T: X \rightarrow L_0$ of the form

$$(1.2) \quad Tf(t) = \int f(s) d\mu_t(s) \quad \text{a.e.}$$

for $f \in X$, where $t \mapsto \mu_t$ is a weak*-Borel map, are called pseudo-integral and have been studied by Arveson [1] and Sourour [18], [19]. Clearly if T has a representation (1.1) then it is pseudo-integral. We note (see Theorem 3.2 of this paper or [11]) that $T: X \rightarrow L_0$ is pseudo-integral if and only if it is regular, i.e. there is a positive operator $P: X \rightarrow L_0$ with $P|f| \geq |Tf|$ for $f \in X$.

In this paper we first study conditions under which (1.1) holds only for $f \in L_\infty$. We show that if X has trivial dual then it is necessary and sufficient that T is *controllable* i.e., there exists $h \in L_0$ so that $|Tf| \leq h$ a.e. for $f \in L_\infty$ with $\|f\|_\infty \leq 1$ (Theorem 4.4 below). It follows from this result that a pseudo-integral operator on X can be represented in the form (1.1.).

Next we seek conditions on X so that every operator is controllable, and we show (Theorem 5.1) that if $X \supset L(1,\infty)$ (weak L_1) or more strictly if X contains the closure of the simple functions in $L(1,\infty)$, then this is the case. A converse is also established for a restricted class of spaces X . In particular if X is the Lorentz