

Convex Functions of a Measure and Applications

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Introduction

Our aim in this article is to study the image of a bounded measure μ by a nonlinear continuous function f which has at most a linear growth at infinity and satisfies some other appropriate assumptions. Given a bounded measure μ on an open set Ω of \mathbf{R}^n , and given f as indicated above, one can define $f(\mu)$ as a bounded measure on Ω , and one can define a functional $F(\mu)$ with values in \mathbf{R} by setting

$$(*) \quad F(\mu) = \int_{\Omega} f(\mu).$$

When $f(\xi) = |\xi|$ is the norm of ξ in X or when $X = \mathbf{R}$ and $f(\xi) = \xi^+$ or ξ^- , we recover the usual measures $f(\mu) = |\mu|$, μ^+ or μ^- ; when μ is absolutely continuous with respect to the Lebesgue measure dx , $\mu = udx$, then simply $f(\mu) = f \circ udx$ and

$$F(\mu) = \int_{\Omega} f(u(x))dx.$$

Such functions were defined and studied in the convex case in H. Brézis [4], C. Goffman and J. Serrin [14], R. T. Rockafellar [21], R. Temam [26] which deal with or emphasize the functional (*). In the present article motivated by the applications to nonlinear mechanics which are briefly described in Section 3, we are interested in the properties of the measure $f(\mu)$ itself and its continuous dependence with respect to μ .

Section 1 contains the definition of the measure $f(\mu)$ and its general properties which are valid as well in the convex and in the nonconvex case. When f is convex some other properties specific (as far as we know) to that case are given, in particular a duality formula which relates the definition of $f(\mu)$ to the concept of convex conjugate functions, i.e. the Legendre transform of f . Section 2 contains approximation results for $f(\mu)$: it is clear that if a sequence of measures μ_j converges weakly to a measure μ , we cannot in general expect $f(\mu_j)$ to converge to $f(\mu)$; however some appropriate convergence and approximation results are given in Section 2. A typical approximation result provides a sequence of smooth func-