

Uniqueness of Unbounded Viscosity Solution of Hamilton-Jacobi Equations

HITOSHI ISHII

§0. Introduction

Recently Crandall and Lions [6] introduced the notion of “viscosity solutions,” a kind of generalized solutions for nonlinear first order partial differential equations. They proved both uniqueness and existence theorems, valid under quite weak hypotheses. Somewhat later, Crandall, Evans and Lions [5] reformulated and simplified this work.

We recall the definition of viscosity solutions. Let \mathcal{O} be a domain of \mathbf{R}^M and $F: \mathcal{O} \times \mathbf{R}^1 \times \mathbf{R}^M \rightarrow \mathbf{R}^1$ be continuous. Consider the first order partial differential equation

$$(0.1) \quad F(y, u, Du) = 0 \quad \text{in } \mathcal{O},$$

where $u: \mathcal{O} \rightarrow \mathbf{R}^1$ is the unknown and Du denotes the gradient

$$\left(\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_M} \right).$$

A function $u \in C(\mathcal{O})$ is called a *viscosity solution* of (0.1) if the following two conditions are satisfied: (i) if $\phi \in C^1(\mathcal{O})$ and $u - \phi$ attains its local maximum at $y_0 \in \mathcal{O}$, then

$$(0.2) \quad F(y_0, u(y_0), D\phi(y_0)) \leq 0,$$

and (ii) if $\phi \in C^1(\mathcal{O})$ and $u - \phi$ attains its local minimum at $y_0 \in \mathcal{O}$, then

$$(0.3) \quad F(y_0, u(y_0), D\phi(y_0)) \geq 0.$$

Our purpose here is mainly to study uniqueness of (possibly) unbounded viscosity solutions of the stationary problem

$$(0.4) \quad u + H(x, Du) = 0 \quad \text{in } \mathbf{R}^N,$$

and the Cauchy problem

$$(0.5) \quad \begin{cases} u_t + H(t, x, Du) = 0 & \text{in } (0, T) \times \mathbf{R}^N, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbf{R}^N. \end{cases}$$