

Differential Games and Representation Formulas for Solutions of Hamilton-Jacobi-Isaacs Equations

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1. Introduction

Recent work by the authors and others has demonstrated the connections between the dynamic programming approach to two-person, zero-sum differential games and the new notion of “viscosity” solutions of Hamilton-Jacobi PDE, introduced in Crandall-Lions [8]. The formal relationships here were observed by Isaacs in the early 1950’s (cf. [18]): he showed that if the values of various differential games are regular enough, then they solve certain first order PDE with “max-min” or “min-max” type nonlinearity (the Isaacs equations). The problem here is that usually the value functions are not sufficiently smooth to make sense of these PDE in any obvious way. Many later papers in the subject have worked around this difficulty: see especially Fleming [13], [14], Friedman [15], [16], Elliott-Kalton [9]–[11], Krassovski-Subbotin [20], Subbotin [26], etc., and the references therein.

Recently, however, M. Crandall and P. L. Lions [8] have discovered a new notion of weak or so-called “viscosity” solution for Hamilton-Jacobi equations, and, most importantly, have proved uniqueness of such a solution in a wide variety of circumstances. This concept was reconsidered and simplified in part by Crandall, Evans, Lions [7], whose approach we follow below. Additionally, Lions in his new book [21] has made the fundamental observation that the dynamic programming optimality condition for the value in differential control theory problems implies that this value function is the viscosity solution of the associated Hamilton-Jacobi-Bellman PDE: see [21, pages 53–54] for more explanation. Some related papers are Lions [23], Lions-Nisio [24], Capuzzo Dolcetta-Evans [5], Barles [2], Capuzzo Dolcetta [4], Capuzzo Dolcetta-Ishii [6], etc.

The foregoing considerations turn out to extend to differential game theory, where additional complications arise with respect to the definition of the value functions. Nevertheless the basic idea is still valid, that the dynamic programming optimality conditions imply that the values are viscosity solutions of appropriate PDE. See Souganidis [27] for a demonstration of this based on both the Fleming and the Friedman definitions of upper and lower values for a differential game,