

On the Smoothness of the Nonlinear Spectral Manifolds Associated to the Navier-Stokes Equations

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Introduction. We consider the Navier-Stokes equations

$$(0.1) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

in $\Omega \times]0, \infty[$, where Ω is a smooth bounded open set in \mathbf{R}^n , $n = 2, 3$, or the cube $]0, L[^n$.

We supplement (0.1) with the boundary condition

$$(0.2) \quad u|_{\partial\Omega} = 0,$$

or with the spatial periodic condition

$$(0.3) \quad u(x + L e_j) = u(x) \quad \forall x \in \mathbf{R}^n, 1 \leq j \leq n$$

where (e_j) , $1 \leq j \leq n$, is the canonical basis in \mathbf{R}^n .

In previous works [3], [4], [5], we investigated the asymptotic behavior of regular (we shall make this term more precise later on) solutions u of (0.1), (0.2) (respectively (0.1), (0.3)) as $t \rightarrow +\infty$. Roughly speaking, $u(t)$ decays *exactly* as $e^{-\nu\Lambda(u_0)t}$ where $\Lambda(u_0)$ is an eigenvalue of the Stokes operator. Moreover, there exists in the space \mathcal{R} of initial data u_0 , a flag of analytic sets M_k , $k = 1, 2, \dots$ such that $\Lambda(u_0) = \Lambda_k$, the k^{th} distinct eigenvalue of Stokes operator, if and only if $u_0 \in M_{k-1} \setminus M_k$ where $M_0 = \mathcal{R}$. It was also shown in [3], [4] that the nonlinear spectral set M_k is a *smooth* analytic manifold of finite codimension around the origin.

In the present paper, we proceed to the study of the smoothness of the M_k 's. We shall prove that M_k is a smooth analytic manifold, i.e. M_k has no singularities.

The proof lies on the study of the asymptotic behavior of some linearized Navier-Stokes equations around a rapidly decaying vectorfield. For these linear equations, we develop an asymptotic theory of solutions similar to that of the nonlinear