

The Cauchy Transform of the Principal Function Associated with a Non-normal Operator

KEVIN F. CLANCEY

Dedicated to the memory of James P. Williams

A bounded linear operator T on a Hilbert space \mathcal{H} is called nearly normal in case the self-commutator $[T^*, T] = T^*T - TT^*$ is trace class. The principal function associated with a nearly normal operator T is a real valued, compactly supported Lebesgue integrable function g_T defined on the complex plane which provides, for all polynomials $p = p(z, \bar{z})$, $q = q(z, \bar{z})$ in the variables z , \bar{z} the representation

$$(1.1) \quad \text{tr}[p(T, T^*), q(T, T^*)] = \frac{1}{\pi} \int_{\mathbb{C}} \{\bar{\partial} p \partial q - \partial p \bar{\partial} q\} g_T dm_2,$$

where m_2 denotes planar Lebesgue measure. The principal function was first introduced in [16], [17]. The tracial bilinear form on the algebra of polynomials in z and \bar{z} defined by (1.1) was first studied in [13], where it was represented as integration of the Jacobian $\bar{\partial} p \partial q - \partial p \bar{\partial} q$ against a signed measure on the plane. The connection of g_T with this bilinear form is due to J. D. Pincus.

It is clear from (1.1) that the principal function does not depend on any normal direct summand of the operator T . When the operator T is pure (equivalently, when T has no non-zero normal direct summands), then the principal function g_T is a fairly fine unitary invariant. In particular, g_T captures most of the spectral properties of T [5], [7], [8], [13], [18], and sometimes can be used in a remarkable manner to measure the extent to which the operator T is (multi-) cyclic [1], [4]. If the self-commutator $[T^*, T]$ of the irreducible operator T is one dimensional, then g_T is a complete unitary invariant [16]. (See also [6], [11].)

Let $M_0(\mathbb{C})$ denote the space of all compactly supported complex Borel measures on the complex plane. Recall that for ν in $M_0(\mathbb{C})$ the Cauchy transform of ν is the locally integrable function $\hat{\nu}$ defined a.e. on \mathbb{C} by

$$\hat{\nu}(z) = \frac{1}{\pi} \int \frac{d\nu(\zeta)}{\zeta - z}.$$