

Lorentz Space Forms and Virtually Solvable Groups

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Dedicated to Professor Nobuo Shimada for his sixtieth birthday

In this paper we shall study the structure of Lorentz space forms whose fundamental groups are virtually solvable. Let V denote a vector space of real $(n + 2)$ -tuples $v = (x_1, x_2, y_1, \dots, y_n)$ with a nondegenerate quadratic form

$$Q(v) = -x_1^2 - x_2^2 + y_1^2 + \dots + y_n^2$$

and $S^{1,n}$ denote the quadric $\{v \in V : Q(v) = -1\}$. $Q|S^{1,n}$ is a nondegenerate complete metric of type $(1, n)$ and constant curvature -1 (cf. [14], [23]). Let $\widetilde{S}^{1,n}$ be the universal covering of $S^{1,n}$ equipped with the metric induced from that of $S^{1,n}$. $\widetilde{S}^{1,n}$ is a simply connected complete pseudo Riemannian manifold of type $(1, n)$ and constant curvature -1 . (It is written \widetilde{H}_1^{n+1} in Wolf's book.) Then the complete pseudo Riemannian manifolds of type $(1, n)$ and constant curvature -1 are just the quotients $\pi \backslash \widetilde{S}^{1,n}$ where π is a group of isometries that act freely and properly discontinuously. By a Lorentz space form we shall mean such a pseudo Riemannian manifold throughout this paper. According to Kulkarni [14], Lorentz space forms contain the set of strict space forms $\{\Gamma \backslash S^{1,n} : \Gamma \subset O(2, n)\}$, and the set of standard space forms $\{\Gamma \backslash S^{1,2n} : \Gamma \subset U(1, n)\}$, where $O(2, n)$ (respectively $U(1, n)$) is the full group of Q -orthogonal (respectively unitary) transformations so that $U(1, n) \subset O(2, 2n)$. When a Lorentz space form $\pi \backslash \widetilde{S}^{1,n}$ supports a certain S^1 action, we observe in §1 that there exists a simply connected complete Riemannian manifold of sectional curvature ≤ -1 on which π/\mathbf{Z} acts properly discontinuously by isometries, where \mathbf{Z} is an infinite central subgroup of π . Then we may use the result of visibility manifolds (cf. [8], [4]) to characterize π/\mathbf{Z} . As an application when $\pi \backslash \widetilde{S}^{1,2n}$ is a standard space form, it is shown in §2 that π/\mathbf{Z} is uniquely determined by an amenable Lie group in $U(1, n)$. An amenable Lie group is an extension of a solvable group by a compact group (cf. [9], [17]).

In this case G is the semidirect product $(\mathbf{R} \times \mathbf{C}^{n-1}) \circ (U(n-1) \times \mathbf{C}^*)$.

Theorem 2.9. *Let $\Gamma \backslash S^{1,2n}$ be a standard space form. If Γ contains a nontrivial normal solvable subgroup, then it is isometric to $g\Gamma g^{-1} \backslash G/U(n-1)$ for some $g \in U(1, n)$. In that case,*