

Harmonic Forms and L^2 Cohomology on Manifolds with Cylinders

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Introduction. In this paper we consider harmonic forms and L^2 cohomology groups on a manifold with cylinders, and discuss relationships between these spaces and the topology of the manifold. Here a manifold with cylinders \hat{M} means a compact oriented Riemannian manifold M with boundary N , where the metric is a product near N , together with a copy of $N \times \mathbf{R}$, carrying the product metric, glued onto N .

The goal of §1 is a new elementary proof of the η -invariant formula of Atiyah, Patodi, and Singer [1] for the Gauss-Bonnet operator on forms. Although it is easy to show that the η -invariant vanishes for this operator, the formula is still not trivial. Our proof hinges on a Hodge Theorem for the space of limiting values of harmonic forms on \hat{M} (Theorem 1.11). This theorem in turn reduces to showing that a harmonic form in $\text{Im}(H^*(M) \rightarrow H^*(N))$ can be extended to a harmonic form on \hat{M} . This extension is a qualitative improvement on Conner's version of the Hodge Theorem for arbitrary Riemannian manifolds with boundary [3], and it reflects the special nature of the metric on \hat{M} . With Theorem 1.11, all the terms in the η -invariant formula (1.5) have a topological interpretation, and the formula becomes an exercise in elementary algebraic topology. Theorem 1.11 also allows us to prove a conjecture in [1] that every harmonic form on N can be written uniquely as a sum of an even dimensional limiting value and an odd dimensional limiting value (Corollary 1.17). The rest of §1 is given to some examples and computations of the dimensions of the various spaces of harmonic forms.

In §2 we first recall the definition and basic properties of L^2 cohomology. We then compute that the i^{th} L^2 cohomology group of \hat{M} is finite dimensional if and only if $H^{i-1}(N) = 0$. This can also be derived from results in [8], but the proofs here are more elementary for the case at hand. Moreover, we obtain some added information about the spectrum of the Laplacian forms on \hat{M} . In particular, we see that while the spaces of L^2 harmonic forms on \hat{M} reflect how N sits in M , the spectrum right above zero depends only on N itself.

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