

Homotopy Classification of Higher Dimensional Links

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1. Introduction. The relation of link homotopy was introduced in classical link theory by Milnor [M]. One allows a link (i.e. disjoint collection of simple closed curves in 3-space) to undergo arbitrary deformations, provided the components are kept disjoint at each stage of the homotopy. The purpose of this paper is to initiate a study of the analogous equivalence relation in higher dimensions. The “links” that we consider are disjoint collections of spheres (of various dimensions) in Euclidean m -space R^m or in the m -sphere S^m , $m \geq 3$.

The only earlier paper related to this subject (to the best of our knowledge) is by P. Scott [S]. However, this paper is concerned with arbitrary maps of spheres into S^m , with the requirement that the images be disjoint. The maps are not required to be imbeddings as in the present paper.

The theory naturally separates into two cases: links with all components of codimension at least three, and links with at least one component of codimension two. (We always assume codimension at least two.) In the former case, the link-homotopy classes of links of a given type form an abelian group, the addition being connected sum. These groups are homomorphic images of the link groups studied by Haefliger [Hf].

Narrowing the study to links of two components, we are able to achieve a classification of the link homotopy groups, under certain dimensional hypotheses. This is achieved using Haefliger’s theory and the “ α -invariant”—a homomorphism from the link group into $\pi_{p_1+p_2}(S^{m-1})$.

The case of codimension two is studied by essentially different techniques, including surgery. Our principal result again is for the two-component case, and classifies such links, up to link homotopy, in roughly half of the cases.

Links of three or more components present more subtle problems, just as in classical dimensions. One of the authors has made some progress in this direction, but the principal results of this paper are confined to the two-component case.

The paper is organized as follows. Basic definitions and preliminary observations are in Section 2. Sections 3 and 4 introduce the α -invariant and the invariants λ_1 and λ_2 for links of two components, and establish some connections between them. Links of codimension greater than two are discussed in Section 5, and the classification theorem for such links is stated and proved in Section 6. Section 7