

An Orbital Perspective on Square-Integrable Representations

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Introduction and terminology. In this paper we shall consider the most general kind of square-integrable representations of Lie groups. In fact there is now a single unified method—the Orbit Method—for describing the harmonic analysis and representation theory of an *arbitrary* Lie group (the method is developed fully in [5]). It is well known—and highlighted by [13]—that the square-integrable representations play a fundamental role in the harmonic analysis of *all* Lie groups. Thus it is important to understand as fully as possible what position those representations occupy in the Orbit Method. We shall be concerned with a host of problems surrounding that idea in this paper. The main results are: Theorem 1—a criterion for square-integrable representations in terms of open orbits; Theorem 3—an orbital realization of an arbitrary square-integrable representation; Theorem 5—which generalizes a key property of square-integrable representations to generic irreducible representations, and is related to [13, Theorem A]; Proposition 6—a simplified construction of square-integrable representations in the special case of H-groups; Proposition 10—a distinguishing property of square-integrable representations. To some extent these constitute a potpourri of results, no one of which is the most important. But I feel it is useful, first—as the title suggests—for placing square-integrable representations into the perspective that has come to dominate the subject, and second as a completion of the circle of ideas developed in [10]-[13].

We establish now the terminology of the paper. G will be a Lie group, \mathfrak{g} its Lie algebra, \mathfrak{g}^* the real linear dual. We write $\mathcal{AP}(G)$ for the set of admissible, well-polarizable elements in \mathfrak{g}^* (see [5], [10]). For each $\phi \in \mathcal{AP}(G)$ we write G_ϕ for the stability group, G_ϕ^0 its identity component, and $\mathcal{X}_G(\phi)$ for the set of irreducible unitary extensions of “ $\exp i\phi$ ” (actually a canonical projective dual of G_ϕ/G_ϕ^0 —see [5], [10]). We set $\mathcal{B}(G) = \{(\phi, \tau) : \phi \in \mathcal{AP}(G), \tau \in \mathcal{X}_G(\phi)\}$. $\mathcal{B}(G)$ is naturally a G -space and $\mathcal{B}(G)/G$ parameterizes a generic set in \hat{G} (see [5]). Now suppose Ξ is a unitary character of the center Z_G of G , $\mathfrak{z} = LA(Z_G)$, and $\xi = (1/i)d\Xi \in \mathfrak{z}^*$. We set $\mathfrak{q}_\xi^* = \{\phi \in \mathfrak{g}^* : \phi|_{\mathfrak{z}} = \xi\}$, $\mathcal{AP}_\xi(G) = \mathcal{AP}(G) \cap \mathfrak{q}_\xi^*$, $\mathcal{X}_{G, \Xi} =$ the part of $\mathcal{X}_G(\phi)$ lying over Ξ (see [5, p. 155]),

$$\mathcal{B}_\Xi(G) = \{(\phi, \tau) \in \mathcal{B}(G) : \phi \in \mathcal{AP}_\xi(G), \tau \in \mathcal{X}_{G, \Xi}(\phi)\}.$$

Then $\mathcal{B}_\Xi(G)/G$ parameterizes a generic set in the spectrum of $\text{Ind}_{Z_G}^G \Xi$. Next we