

On the Minimum Area Problem

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1. Introduction. Let S denote the class of univalent functions f in $\Delta = \{z: |z| < 1\}$ with $f(0) = 0, f'(0) = 1$. In [3] Goodman proposed the following problem: Find

$$(1.1) \quad \inf_{f \in S} \{\text{area}[f(\Delta) \cap \Delta]\} = \beta_1.$$

Using a normal family type argument, it is easily seen that there exists $f_1 \in S$ with

$$(1.2) \quad \text{area}[f_1(\Delta) \cap \Delta] = \beta_1.$$

Barnard and Suffridge (see [2, p. 536]) have made the following observations concerning f_1 :

(1.3) f_1 is circularly symmetric with respect to a ray through $z = 0$. If this ray is chosen to be the positive real axis, then $\mathbf{C} - [f_1(\Delta) \cup \Delta]$ consists of $(-\infty, -1)$ and an arc of $\{z: |z| = 1\}$, which is symmetric about the real axis and contains -1 .

The first sentence in (1.3) is a consequence of the fact that circular symmetrization preserves area and increases the mapping radius of a simply-connected domain [7], as well as the observation that if $g'(0) > 1, g/g'(0) \in S$, and

$$(1.4) \quad \text{area}[g(\Delta) \cap \Delta] = \text{area}[f(\Delta) \cap \Delta],$$

where $f \in S$, then f cannot satisfy (1.2) with f_1 replaced by f (see the discussion following (6.3)). The second sentence in (1.3) follows from the first, (1.4), and simple properties of subordination.

Let $\bar{E}, \partial E$ denote the closure and boundary of the set E . Put $\gamma = (\partial f_1(\Delta)) \cap \Delta$. The above authors also state that if γ is assumed twice continuously differentiable, then it can be shown that $|f_1'|$ is constant on $f_1^{-1}(\gamma)$. In this paper, we prove

Theorem 1. *Let $f_1 \in S$ satisfy (1.2) and (1.3). There exist $\theta_0, \theta_1, 0 < \theta_0 < \theta_1 < \pi$, and $\alpha > 0$ such that if $E_i = \{e^{i\theta} : \theta_i < \theta < 2\pi - \theta_i\}, i = 0, 1$, then*

(1.5) f_1' has a nonzero continuous extension to $E_0 \cup \Delta$ which is Hölder continuous with exponent $1/2$ on E_0 ,