

Power Concavity and Boundary Value Problems

ALAN U. KENNINGTON

1. Introduction. This article presents an improved version of Korevaar's convexity maximum principle (1983, [4]), which is used to show that positive powers of solutions of various categories of boundary value problems are concave. For instance, it is shown that if $\Delta u + f(x) = 0$ for x in a bounded convex domain Ω in \mathbf{R}^n for some $n \geq 2$, f is non-negative, f^β is concave in Ω for some $\beta \geq 1$, and $u = 0$ on $\partial\Omega$, then u^α is concave in Ω for $0 < \alpha \leq \beta/(1 + 2\beta)$. The upper bound for α is shown to be sharp.

More generally, it is shown for any $\alpha \in (0,1]$ that if u is a solution of $\Delta u + b(x,u) = 0$ in Ω such that $u = 0$ on $\partial\Omega$, then u^α is concave in $\bar{\Omega}$ whenever $b(x,u) \geq 0$ for all x and u , $t^{\alpha-1}b(x,t)$ is decreasing with respect to t , and $t^{(3\alpha-1)/\alpha}b(x,t^{1/\alpha})$ is jointly concave with respect to (x,t) . Similar results are obtained for the equation $\Delta u = e^u$.

2. Definitions and preliminary results. Let Ω be a domain in \mathbf{R}^n , $n \geq 2$. For k a non-negative integer or $+\infty$, $C(\Omega)$, $C(\bar{\Omega})$, $C^k(\Omega)$ and $C^k(\bar{\Omega})$ will denote respectively the set of continuous functions on Ω , the set of continuous functions on $\bar{\Omega}$, the set of functions in $C(\Omega)$ whose derivatives of order less than or equal to k are continuous, and the set of functions in $C(\bar{\Omega})$ whose derivatives in Ω of order less than or equal to k have continuous extensions to $\bar{\Omega}$.

If Ω is a bounded convex domain in \mathbf{R}^n and $u : \bar{\Omega} \rightarrow \mathbf{R}$ is a bounded function, then the *convexity function* c for u on $\bar{\Omega}$ is defined on $\bar{\Omega} \times \bar{\Omega} \times [0,1]$ by:

$$c(y,z,\lambda) = (1 - \lambda)u(y) + \lambda u(z) - u((1 - \lambda)y + \lambda z) \quad \text{for } y, z \in \bar{\Omega}, \lambda \in [0,1].$$

Write $\bar{c} = \sup\{c(y,z,\lambda) : (y,z,\lambda) \in \bar{\Omega} \times \bar{\Omega} \times [0,1]\}$. Then \bar{c} is a real number since c is bounded. Also, $\bar{c} \geq 0$ (which follows, for instance, by putting $\lambda = 0$), and $\bar{c} = 0$ if and only if u is concave in $\bar{\Omega}$.

In order to present the results of this paper coherently, it is convenient to introduce the concept of α -concavity. When α is a positive real number, a non-negative function u defined on a convex subset of \mathbf{R}^n for any $n \geq 1$ is said to be α -concave when its α^{th} power, u^α , is concave. Then α -concavity can be extended in a natural way to all extended real numbers α in $[-\infty, +\infty]$ as follows: u is said to be α -concave for $\alpha = +\infty$ when u is constant, for $0 < \alpha < +\infty$ when u^α is